Aberystwyth University

Two-parameter anisotropic homogenization for a Dirichlet problem for the Poisson equation in an unbounded periodically perforated domain

de Cristoforis, Massimo Lanza; Musolino, Paolo

Published in:
Mathematische Nachrichten

DOI:
10.1002/mana.201600414

Publication date:
2018

Citation for published version (APA):
de Cristoforis, M. L., & Musolino, P. (2018). Two-parameter anisotropic homogenization for a Dirichlet problem for the Poisson equation in an unbounded periodically perforated domain: A functional analytic approach. *Mathematische Nachrichten*, *291*(8-9), 1310-1341. https://doi.org/10.1002/mana.201600414

Two-parameter anisotropic homogenization for a Dirichlet problem for the Poisson equation in an unbounded periodically perforated domain. A functional analytic approach

Massimo Lanza de Cristoforis * Paolo Musolino†

Abstract: We consider a Dirichlet problem for the Poisson equation in an unbounded periodically perforated domain. The domain has a periodic structure, and the size of each cell is determined by a positive parameter δ, and the level of anisotropy of the cell is determined by a diagonal matrix γ with positive diagonal entries. The relative size of each periodic perforation is instead determined by a positive parameter ϵ. For a given value $\tilde{\gamma}$ of γ, we analyze the behavior of the unique solution of the problem as $(\epsilon, \delta, \gamma)$ tends to $(0, 0, \tilde{\gamma})$ by an approach which is alternative to that of asymptotic expansions and of classical homogenization theory.

Keywords: Dirichlet problem, singularly perturbed domain, Poisson equation, periodically perforated domain, integral equations, anisotropic homogenization, real analytic continuation in Banach space

2010 Mathematics Subject Classification: 35J25; 31B10; 45A05; 47H30

1 Introduction

In this paper, we consider a Dirichlet problem for the Poisson equation in a periodically perforated domain with small holes. We fix once for all

$$n \in \mathbb{N} \setminus \{0, 1\}, \qquad \text{and} \qquad (q_{11}, \ldots, q_{nn}) \in]0, +\infty[^n,$$

and we introduce a periodicity cell

$$Q \equiv \Pi_{j=1}^n]0, q_{jj}[\,.$$

Then we denote by q the diagonal matrix

$$q \equiv \begin{pmatrix} q_{11} & 0 & \ldots & 0 \\ 0 & q_{22} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & q_{nn} \end{pmatrix}$$

and by $m_n(Q)$ the n dimensional measure of the fundamental cell Q. Clearly, $q\mathbb{Z}^n \equiv \{qz : z \in \mathbb{Z}^n\}$ is the set of vertices of a periodic subdivision of $\mathbb{R}^n$ corresponding to the fundamental cell Q.

Then we consider $m \in \mathbb{N} \setminus \{0\}$ and $\alpha \in]0, 1[$ and a subset Ω of $\mathbb{R}^n$ satisfying the following assumption.

$$\begin{aligned} &\text{Let } \Omega \text{ be a bounded open connected subset of } \mathbb{R}^n \text{ of class } C^{m,\alpha}. \\ &\text{Let } \mathbb{R}^n \setminus \mathrm{cl}\Omega \text{ be connected. Let } 0 \in \Omega\,. \end{aligned} \tag{1}$$

Next we fix $p \in Q$. Then there exists $\epsilon_0 \in]0, +\infty[$ such that

$$p + \epsilon \mathrm{cl}\Omega \subseteq Q \qquad \forall \epsilon \in] - \epsilon_0, \epsilon_0[\,, \tag{2}$$

where cl denotes the closure. To shorten our notation, we set

$$\Omega_{p,\epsilon} \equiv p + \epsilon\Omega \qquad \forall \epsilon \in \mathbb{R}\,.$$

*Dipartimento di Matematica, Università degli Studi di Padova, via Trieste 63, 35121 Padova, Italy
†Department of Mathematics, Aberystwyth University, Ceredigion SY23 3BZ, Wales, UK.

Then we introduce the periodic domains

$$\mathbb{S}[\Omega_{p,\epsilon}] \equiv \bigcup_{z\in\mathbb{Z}^n} (qz + \Omega_{p,\epsilon}) \,, \qquad \mathbb{S}[\Omega_{p,\epsilon}]^- \equiv \mathbb{R}^n \setminus \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}] \,,$$

for all $\epsilon \in]-\epsilon_0, \epsilon_0[$. Then a function u from $\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]$ or from $\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-$ to $\mathbb{C}$ is q-periodic if $u(x+q_{hh}e_h) = u(x)$ for all x in the domain of u and for all $h \in \{1,\dots,n\}$. Here $\{e_1,\dots, e_n\}$ denotes the canonical basis of $\mathbb{R}^n$.

Next we introduce a two-parameter anisotropic version of the periodic domain $\mathbb{S}[\Omega_{p,\epsilon}]^-$. To do so, we introduce an anisotropic dilation in the form of a diagonal matrix. Let $\mathbb{D}_n(\mathbb{R})$ denote the space of $n \times n$ diagonal matrices with real entries. Let $\mathbb{D}_n^+(\mathbb{R})$ be the set of elements of $\mathbb{D}_n(\mathbb{R})$ with diagonal entries in $]0,+\infty[$. Then we set

$$\mathbb{S}(\epsilon,\delta,\gamma)^- \equiv \delta\gamma\mathbb{S}[\Omega_{p,\epsilon}]^- \qquad \forall(\epsilon,\delta,\gamma) \in]0,\epsilon_0[\times]0,+\infty[\times\mathbb{D}_n^+(\mathbb{R}) \,.$$

The parameter δ determines the size of the periodic cells of $\mathbb{S}(\epsilon,\delta,\gamma)^-$, while the matrix γ determines the anisotropy. In particular, if γ equals a positive multiple of the identity matrix I, then we have isotropic homogenization.

Next we turn to introduce the data of our problem. Let

$$f \text{ be a } q-\text{periodic real analytic function from } \mathbb{R}^n \text{ to } \mathbb{R} \text{ such that } \int_Q f\,dx = 0 \,. \tag{3}$$

Then we assign a function g in the Schauder class $C^{m,\alpha}(\partial\Omega)$, and we consider the Dirichlet problem

$$\begin{cases} \Delta u(x) = f(\delta^{-1}\gamma^{-1}x) & \forall x \in \mathbb{S}(\epsilon,\delta,\gamma)^- \,, \\ u \text{ is } \delta\gamma q-\text{periodic in } \mathbb{S}(\epsilon,\delta,\gamma)^- \,, & \\ u(x) = g(\delta^{-1}\gamma^{-1}\epsilon^{-1}(x-\delta\gamma p)) & \forall x \in \delta\gamma\partial\Omega_{p,\epsilon} \,. \end{cases} \tag{4}$$

If $(\epsilon,\delta,\gamma) \in]0,\epsilon_0[\times]0,+\infty[\times\mathbb{D}_n^+(\mathbb{R})$ and if f and g are as above, then it is well known that there exists a unique solution

$$u(\epsilon,\delta,\gamma,\cdot) \in C^{m,\alpha}(\mathrm{cl}\mathbb{S}(\epsilon,\delta,\gamma)^-) \,.$$

Now let $\tilde{\gamma} \in \mathbb{D}_n^+(\mathbb{R})$. The goal of this paper is to investigate the behavior of $u(\epsilon,\delta,\gamma,\cdot)$ and of the energy integral

$$\mathrm{En}[\epsilon,\delta,\gamma] \equiv \int_{Q\cap\mathbb{S}(\epsilon,\delta,\gamma)^-} |D_x u(\epsilon,\delta,\gamma,x)|^2\,dx$$

of the solution in the cell Q as (ϵ,δ,γ) tends to $(0,0,\tilde{\gamma})$. In particular, we pose the following questions.

(j) What can we say on the function $(\epsilon,\delta,\gamma) \mapsto u(\epsilon,\delta,\gamma,\cdot)$ as (ϵ,δ,γ) degenerates to the triple $(0,0,\tilde{\gamma})$ in $]0,\epsilon_0[\times]0,+\infty[\times\mathbb{D}_n^+(\mathbb{R})$?

(jj) What can we say on the function $(\epsilon,\delta,\gamma) \mapsto \mathrm{En}[\epsilon,\delta,\gamma]$ as (ϵ,δ,γ) degenerates to the triple $(0,0,\tilde{\gamma})$ in $]0,\epsilon_0[\times]0,+\infty[\times\mathbb{D}_n^+(\mathbb{R})$?

The asymptotic behavior of solutions of problems in periodically perforated domains has long been investigated in the frame of Homogenization Theory. It is perhaps difficult to provide a complete list of contributions, and here we mention, *e.g.*, Ansini and Braides [4], Cioranescu and Murat [10, 11], Conca, Gómez, Lobo and Pérez [12]. We also mention Marčenko and Khruslov [39], and Maz'ya and Movchan [40], where the assumption of periodicity of the array of inclusions has been released.

More generally, problems in singularly perturbed domains have been largely studied with the methods of Asymptotic Analysis and of Calculus of Variations. Here, we mention, *e.g.*, Ammari and Kang [3, Ch. 5], Bonnaillie-Noël, Dambrine, Tordeux, and Vial [5], Dal Maso and Murat [13], Kozlov, Maz'ya, and Movchan [23], Maz'ya, Movchan, and Nieves [41], Maz'ya, Nazarov, and Plamenewskij [42, 43], Nazarov and Sokołowski [48], Ozawa [49], Ward and Keller [52].

We also observe that boundary value problems in domains with periodic inclusions can be analyzed, at least for the two dimensional case, with the method of functional equations. Here we mention, *e.g.*, Castro, Kapanadze, and Pesetskaya [7, 8], Castro, Pesetskaya, and Rogosin [9], Drygas and Mityushev [20], Mityushev and Adler [45], Rogosin, Dubatovskaya, and Pesetskaya [51].

Problems of this type are relevant in the applications. For example, Ammari, Garnier, Giovangigli, Jing, Seo [1] consider transmission problems to study the effective admittivity of cell suspensions. In particular, they use homogenization techniques with asymptotic expansions to derive a homogenized problem and prove two-scale convergence. Moreover, they exploit layer potential techniques to expand the effective admittivity in terms of cell volume fraction for dilute cell suspensions. Analogous techniques have been exploited by Ammari, Giovangigli, Kwon, Seo, and Wintz [2] in order to provide a mathematical scheme which allows to determine microscopic properties of cell cultures from spectral measurements of the effective conductivity.

Here instead, we wish to represent the functions in (j), (jj) in terms of real analytic maps of $(\epsilon, \delta, \gamma)$ and in terms of possibly singular at $\epsilon = 0$, $\delta = 0$, but known functions of ϵ, δ.

This paper is a first step in the analysis of multi-parameter homogenization problems by exploiting a point of view which has already been developed for singular perturbation problems in domains with small periodic holes (cf. *e.g.*, [25, 26, 29, 27].) In the frame of linearized elastostatics and of the Stokes equations, we mention [15, 16] and [14], and for periodic problems we refer to [32, 46, 47].

Our approach is based on potential theory and, more precisely, on periodic layer potentials built by replacing the fundamental solution of the Laplace operator by a γq-periodic analog of the fundamental solution, *i.e.*, a γq-periodic locally integrable function $S_{\gamma q,n}$ such that

$$\Delta S_{\gamma q,n} = \sum_{z\in\mathbb{Z}^n} \delta_{\gamma q z} - \frac{1}{m_n(\gamma Q)}\,,$$

in the sense of distributions in $\mathbb{R}^n$, where $\delta_{\gamma q z}$ denotes the Dirac measure with mass in $\gamma q z$. As is well known, such a γq-periodic analog of the fundamental solution exists and is determined up to an additive constant (cf. *e.g.*, Ammari and Kang [3, p. 53], [30, §3].) The distribution $S_{\gamma q,n}$ is determined up to an additive constant, and we take

$$S_{\gamma q,n}(x) = -\sum_{z\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{m_n(\gamma Q)4\pi^2|(\gamma q)^{-1}z|^2} e^{2\pi i((\gamma q)^{-1}z)\cdot x}\,,$$

in the sense of distributions in $\mathbb{R}^n$ (cf. *e.g.*, Ammari and Kang [3, p. 53], [30, §3].)

Thus, if we want to investigate the dependence of $u(\epsilon,\delta,\gamma,\cdot)$ upon (ϵ,δ,γ) by a potential theoretic approach, we also need to know the regularity properties of the family $\{S_{\gamma q,n}\}_{\gamma\in\mathbb{D}_n^+(\mathbb{R})}$ of γq-periodic analogs of the fundamental solution of the Laplace operator upon γ. By [33], we know that

> the family $\{S_{\gamma q,n}\}_{\gamma\in\mathbb{D}_n^+(\mathbb{R})}$ of γq-periodic analogs of the fundamental solution is such that the map from $\mathbb{D}_n^+(\mathbb{R})\times(\mathbb{R}^n\setminus q\mathbb{Z}^n)$ to $\mathbb{R}$ which takes (γ,x) to $S_{\gamma q,n}(\gamma x)$ is real analytic. (5)

We note that in case $n = 2$ the existence of a family of periodic analogs of a fundamental solution for which the analyticity property of (5) holds can also be deduced by constructive formulas in terms of special functions as those of Lin and Wang [37], of Mamode [38], and of Mityushev and Adler [45]. For related results, see also [33], whose techniques are completely different from those of [37, 45, 38].

Finally, we observe that in case we fix $\gamma\in\mathbb{D}_n^+(\mathbb{R})$ and we are interested in studying the dependence of the solution of problem (4) only on ϵ and δ, then the proofs of the present paper notably simplify (see [34].)

2 Preliminaries and notation

We denote the norm on a normed space $\mathcal{X}$ by $\|\cdot\|_{\mathcal{X}}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. We endow the space $\mathcal{X}\times\mathcal{Y}$ with the norm defined by $\|(x,y)\|_{\mathcal{X}\times\mathcal{Y}}\equiv\|x\|_{\mathcal{X}}+\|y\|_{\mathcal{Y}}$ for all $(x,y)\in\mathcal{X}\times\mathcal{Y}$, while we use the Euclidean norm for $\mathbb{R}^n$. For standard definitions of Calculus in normed spaces, we refer to Deimling [19]. The symbol $\mathbb{N}$ denotes the set of natural numbers including 0. Let A be a matrix. Then A_{ij} denotes the (i,j)-entry of A and A^t denotes the transpose matrix of A. If A is invertible, A^{-1} denotes the inverse matrix of A. Let $\mathbb{D}\subseteq\mathbb{R}^n$. Then $\mathrm{cl}\mathbb{D}$ denotes the closure of $\mathbb{D}$ and $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$. We also set

$$\mathbb{D}^-\equiv\mathbb{R}^n\setminus\mathrm{cl}\mathbb{D}\,.$$

For all $R>0$, $x\in\mathbb{R}^n$, x_j denotes the j-th coordinate of x, $|x|$ denotes the Euclidean modulus of x in $\mathbb{R}^n$, and $\mathbb{B}_n(x,R)$ denotes the ball $\{y\in\mathbb{R}^n : |x-y|<R\}$. Let Ω be an open subset of $\mathbb{R}^n$. The space of m times continuously differentiable real-valued functions on Ω is denoted by $C^m(\Omega,\mathbb{R})$, or more simply by $C^m(\Omega)$.

Let $r\in\mathbb{N}\setminus\{0\}$. Let $f\in(C^m(\Omega))^r$. The s-th component of f is denoted f_s, and Df denotes the Jacobian matrix $\left(\frac{\partial f_s}{\partial x_l}\right)_{\substack{s=1,\dots,r,\\ l=1,\dots,n}}$. Let $\eta\equiv(\eta_1,\dots,\eta_n)\in\mathbb{N}^n$, $|\eta|\equiv\eta_1+\dots+\eta_n$. Then $D^\eta f$ denotes $\frac{\partial^{|\eta|}f}{\partial x_1^{\eta_1}\dots\partial x_n^{\eta_n}}$. The subspace of $C^m(\Omega)$ of those functions f whose derivatives $D^\eta f$ of order $|\eta|\le m$ can be extended with continuity to $\mathrm{cl}\Omega$ is denoted $C^m(\mathrm{cl}\Omega)$. The subspace of $C^m(\mathrm{cl}\Omega)$ whose functions have m-th order derivatives that are Hölder continuous with exponent $\alpha \in]0,1]$ is denoted $C^{m,\alpha}(\mathrm{cl}\Omega)$ (cf. *e.g.*, Gilbarg and Trudinger [22].) The subspace of $C^m(\mathrm{cl}\Omega)$ of those functions f such that $f_{|\mathrm{cl}(\Omega\cap\mathbb{B}_n(0,R))} \in C^{m,\alpha}(\mathrm{cl}(\Omega\cap\mathbb{B}_n(0,R)))$ for all $R \in]0,+\infty[$ is denoted $C^{m,\alpha}_{\mathrm{loc}}(\mathrm{cl}\Omega)$. If $\mathbb{D} \subseteq \mathbb{R}^r$, we set $C^{m,\alpha}(\mathrm{cl}\Omega,\mathbb{D}) \equiv \{f \in (C^{m,\alpha}(\mathrm{cl}\Omega))^r : f(\mathrm{cl}\Omega) \subseteq \mathbb{D}\}$.

We say that a bounded open subset Ω of $\mathbb{R}^n$ is of class C^m or of class $C^{m,\alpha}$, if $\mathrm{cl}\Omega$ is a manifold with boundary imbedded in $\mathbb{R}^n$ of class C^m or $C^{m,\alpha}$, respectively (cf. *e.g.*, Gilbarg and Trudinger [22, §6.2].) We denote by ν_Ω the outward unit normal to $\partial\Omega$, where it exists. For standard properties of functions in Schauder spaces, we refer the reader to Gilbarg and Trudinger [22] (see also [24, §2, Lem. 3.1, 4.26, Thm. 4.28], [35, §2].)

If M is a manifold imbedded in $\mathbb{R}^n$ of class $C^{m,\alpha}$, with $m \geq 1$, $\alpha \in]0,1[$, one can define the Schauder spaces also on M by exploiting the local parametrizations. In particular, one can consider the space $C^{k,\alpha}(\partial\Omega)$ on $\partial\Omega$ for $0 \leq k \leq m$ with Ω a bounded open set of class $C^{m,\alpha}$, and the trace operator from $C^{k,\alpha}(\mathrm{cl}\Omega)$ to $C^{k,\alpha}(\partial\Omega)$ is linear and continuous. We denote by $d\sigma$ the area element of a manifold imbedded in $\mathbb{R}^n$. We retain the standard notation for the Lebesgue space $L^p(M)$ of p-summable functions. Also, if $\mathcal{X}$ is a vector subspace of $L^1(M)$, we find convenient to set

$$\mathcal{X}_0 \equiv \left\{ f \in \mathcal{X} : \int_M f\, d\sigma = 0 \right\}.$$

We note that throughout the paper 'analytic' means always 'real analytic'. For the definition and properties of analytic operators, we refer to Deimling [19, §15].

We set $\delta_{i,j} = 1$ if $i = j$, $\delta_{i,j} = 0$ if $i \neq j$ for all $i,j = 1,\dots,n$.

If Ω is an arbitrary open subset of $\mathbb{R}^n$, $k \in \mathbb{N}$, $\beta \in]0,1]$, we set

$$C^k_b(\mathrm{cl}\Omega) \equiv \{u \in C^k(\mathrm{cl}\Omega) : D^\gamma u \text{ is bounded } \forall \gamma \in \mathbb{N}^n \text{ such that } |\gamma| \leq k\},$$

and we endow $C^k_b(\mathrm{cl}\Omega)$ with its usual norm

$$\|u\|_{C^k_b(\mathrm{cl}\Omega)} \equiv \sum_{|\gamma|\leq k} \sup_{x\in\mathrm{cl}\Omega} |D^\gamma u(x)| \qquad \forall u \in C^k_b(\mathrm{cl}\Omega).$$

Then we set

$$C^{k,\beta}_b(\mathrm{cl}\Omega) \equiv \{u \in C^{k,\beta}(\mathrm{cl}\Omega) : D^\gamma u \text{ is bounded } \forall \gamma \in \mathbb{N}^n \text{ such that } |\gamma| \leq k\},$$

and we endow $C^{k,\beta}_b(\mathrm{cl}\Omega)$ with its usual norm

$$\|u\|_{C^{k,\beta}_b(\mathrm{cl}\Omega)} \equiv \sum_{|\gamma|\leq k} \sup_{x\in\mathrm{cl}\Omega} |D^\gamma u(x)| + \sum_{|\gamma|=k} |D^\gamma u : \mathrm{cl}\Omega|_\beta \qquad \forall u \in C^{k,\beta}_b(\mathrm{cl}\Omega),$$

where $|D^\gamma u : \mathrm{cl}\Omega|_\beta$ denotes the β-Hölder constant of $D^\gamma u$.

Next, we turn to introduce the Roumieu classes. For all bounded open subsets Ω of $\mathbb{R}^n$ and $\rho > 0$, we set

$$C^0_{\omega,\rho}(\mathrm{cl}\Omega) \equiv \left\{ u \in C^\infty(\mathrm{cl}\Omega) : \sup_{\beta\in\mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\mathrm{cl}\Omega)} < +\infty \right\},$$

and

$$\|u\|_{C^0_{\omega,\rho}(\mathrm{cl}\Omega)} \equiv \sup_{\beta\in\mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\mathrm{cl}\Omega)} \qquad \forall u \in C^0_{\omega,\rho}(\mathrm{cl}\Omega),$$

where $|\beta| \equiv \beta_1 + \cdots + \beta_n$ for all $\beta \equiv (\beta_1,\dots,\beta_n) \in \mathbb{N}^n$. The Roumieu class $\left(C^0_{\omega,\rho}(\mathrm{cl}\Omega), \|\cdot\|_{C^0_{\omega,\rho}(\mathrm{cl}\Omega)}\right)$ is well-known to be a Banach space.

Next we turn to periodic domains. If Ω is an arbitrary subset of $\mathbb{R}^n$ such that $\mathrm{cl}\Omega \subseteq Q$, then we set

$$\mathbb{S}[\Omega] \equiv \bigcup_{z\in\mathbb{Z}^n} (qz + \Omega) = q\mathbb{Z}^n + \Omega, \qquad \mathbb{S}[\Omega]^- \equiv \mathbb{R}^n \setminus \mathrm{cl}\mathbb{S}[\Omega].$$

If Ω is an open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\Omega \subseteq Q$ and if $k \in \mathbb{N}$, $\beta \in]0,1]$, then we set

$$C^k_q(\mathrm{cl}\mathbb{S}[\Omega]) \equiv \left\{ u \in C^k_b(\mathrm{cl}\mathbb{S}[\Omega]) : u \text{ is } q-\text{periodic} \right\},$$

which we regard as a Banach subspace of $C^k_b(\mathrm{cl}\mathbb{S}[\Omega])$, and

$$C^{k,\beta}_q(\mathrm{cl}\mathbb{S}[\Omega]) \equiv \left\{ u \in C^{k,\beta}_b(\mathrm{cl}\mathbb{S}[\Omega]) : u \text{ is } q-\text{periodic} \right\},$$

which we regard as a Banach subspace of $C^{k,\beta}_b(\mathrm{cl}\mathbb{S}[\Omega])$, and

$$C^k_q(\mathrm{cl}\mathbb{S}[\Omega]^-) \equiv \left\{u \in C^k_b(\mathrm{cl}\mathbb{S}[\Omega]^-) : \, u \text{ is } q-\text{periodic}\right\} ,$$

which we regard as a Banach subspace of $C^k_b(\mathrm{cl}\mathbb{S}[\Omega]^-)$, and

$$C^{k,\beta}_q(\mathrm{cl}\mathbb{S}[\Omega]^-) \equiv \left\{u \in C^{k,\beta}_b(\mathrm{cl}\mathbb{S}[\Omega]^-) : \, u \text{ is } q-\text{periodic}\right\} ,$$

which we regard as a Banach subspace of $C^{k,\beta}_b(\mathrm{cl}\mathbb{S}[\Omega]^-)$.

If $\rho \in]0,+\infty[$, then we set

$$C^0_{q,\omega,\rho}(\mathbb{R}^n) \equiv \left\{u \in C^\infty_q(\mathbb{R}^n) : \, \sup_{\beta\in\mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\mathrm{cl}Q)} < +\infty\right\},$$

where $C^\infty_q(\mathbb{R}^n)$ denotes the set of q-periodic functions of $C^\infty(\mathbb{R}^n)$, and

$$\|u\|_{C^0_{q,\omega,\rho}(\mathbb{R}^n)} \equiv \sup_{\beta\in\mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\mathrm{cl}Q)} \qquad \forall u \in C^0_{q,\omega,\rho}(\mathbb{R}^n) .$$

The Roumieu class $\left(C^0_{q,\omega,\rho}(\mathbb{R}^n), \|\cdot\|_{C^0_{q,\omega,\rho}(\mathbb{R}^n)}\right)$ is a Banach space. As is well known, if f is a q-periodic real analytic function from $\mathbb{R}^n$ to $\mathbb{R}$, then there exists $\rho \in]0,+\infty[$ such that

$$f \in C^0_{q,\omega,\rho}(\mathbb{R}^n) . \tag{6}$$

Let $\gamma \in \mathbb{D}^+_n(\mathbb{R})$. Then, $S_{\gamma q,n}$ is even and real analytic in $\mathbb{R}^n \setminus \gamma q\mathbb{Z}^n$ (cf. *e.g.*, Ammari and Kang [3, p. 53], [30, §3].)

Let S_n be the function from $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{R}$ defined by

$$S_n(x) \equiv \begin{cases} \frac{1}{s_n}\log|x| & \forall x \in \mathbb{R}^n\setminus\{0\}, \quad \text{if } n=2 , \\ \frac{1}{(2-n)s_n}|x|^{2-n} & \forall x \in \mathbb{R}^n\setminus\{0\}, \quad \text{if } n>2 , \end{cases}$$

where s_n denotes the $(n-1)$ dimensional measure of $\partial\mathbb{B}_n$. S_n is well-known to be the fundamental solution of the Laplace operator. Then the function $S_{\gamma q,n} - S_n$ can be extended to an analytic function in $(\mathbb{R}^n \setminus \gamma q\mathbb{Z}^n) \cup \{0\}$ (cf. *e.g.*, Ammari and Kang [3, Lemma 2.39, p. 54].) We denote such an extension of $S_{\gamma q,n} - S_n$ by the symbol $R_{\gamma q,n}$ for all $\gamma \in \mathbb{D}^+_n(\mathbb{R})$.

Obviously, $R_{\gamma q,n}$ is not a γq-periodic function. We note that the following elementary equality holds

$$S_{\gamma q,n}(\epsilon x) = \epsilon^{2-n} S_n(x) + \frac{1}{2\pi}(\delta_{2,n}\log\epsilon) + R_{\gamma q,n}(\epsilon x) ,$$

for all $x \in \mathbb{R}^n \setminus \epsilon^{-1}\gamma q\mathbb{Z}^n$ and $\epsilon \in]0,+\infty[$. If Ω is a bounded open subset of $\mathbb{R}^n$ and $f \in L^\infty(\Omega)$, then we set

$$P_n[\Omega, f](x) \equiv \int_\Omega S_n(x-y) f(y)\, dy \qquad \forall x \in \mathbb{R}^n .$$

If we further assume that $\Omega \subseteq \gamma Q$, then we set

$$\mathbb{S}_\gamma[\Omega] \equiv \bigcup_{z\in\mathbb{Z}^n} (\gamma q z + \Omega) = \gamma q\mathbb{Z}^n + \Omega , \qquad \mathbb{S}_\gamma[\Omega]^- \equiv \mathbb{R}^n \setminus \mathrm{cl}\mathbb{S}_\gamma[\Omega] ,$$

and

$$P_{\gamma q,n}[\Omega, f](x) \equiv \int_\Omega S_{\gamma q,n}(x-y) f(y)\, dy \qquad \forall x \in \mathbb{R}^n .$$

Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$ for some $\alpha \in]0,1[$. If H is any of the functions $S_{\gamma q,n}$, $R_{\gamma q,n}$ and $\mathrm{cl}\Omega \subseteq \gamma Q$ or if H equals S_n, we set

$$\begin{aligned} v[\partial\Omega, H, \mu](x) &\equiv \int_{\partial\Omega} H(x-y)\mu(y)\, d\sigma_y \;\forall x \in \mathbb{R}^n , \\ w[\partial\Omega, H, \mu](x) &\equiv \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(y)} H(x-y)\mu(y)\, d\sigma_y = -\int_{\partial\Omega} \nu_\Omega(y) DH(x-y)\mu(y)\, d\sigma_y \;\forall x \in \mathbb{R}^n , \end{aligned}$$

$$w_*[\partial\Omega, H, \mu](x) \quad \equiv \quad \int_{\partial\Omega} \frac{\partial}{\partial \nu_\Omega(x)} H(x-y)\mu(y)\, d\sigma_y = \int_{\partial\Omega} \nu_\Omega(x) DH(x-y)\mu(y)\, d\sigma_y \; \forall x \in \partial\Omega\,,$$

for all $\mu \in L^2(\partial\Omega)$. As is well known, if $\mu \in C^0(\partial\Omega)$, then $v[\partial\Omega, S_{\gamma q,n}, \mu]$ and $v[\partial\Omega, S_n, \mu]$ are continuous in $\mathbb{R}^n$, and we set

$$\begin{aligned} v^+[\partial\Omega, S_{\gamma q,n}, \mu] &\equiv v[\partial\Omega, S_{\gamma q,n}, \mu]_{|\mathrm{cl}\mathbb{S}_\gamma[\Omega]} \qquad & v^-[\partial\Omega, S_{\gamma q,n}, \mu] &\equiv v[\partial\Omega, S_{\gamma q,n}, \mu]_{|\mathrm{cl}\mathbb{S}_\gamma[\Omega]^-} \\ v^+[\partial\Omega, S_n, \mu] &\equiv v[\partial\Omega, S_n, \mu]_{|\mathrm{cl}\Omega} \qquad & v^-[\partial\Omega, S_n, \mu] &\equiv v[\partial\Omega, S_n, \mu]_{|\mathrm{cl}\Omega^-}\,. \end{aligned}$$

Also, if μ is continuous, then $w[\partial\Omega, S_{\gamma q,n}, \mu]_{|\mathbb{S}_\gamma[\Omega]}$ admits a continuous extension to $\mathrm{cl}\mathbb{S}_\gamma[\Omega]$, which we denote by $w^+[\partial\Omega, S_{\gamma q,n}, \mu]$ and $w[\partial\Omega, S_{\gamma q,n}, \mu]_{|\mathbb{S}_\gamma[\Omega]^-}$ admits a continuous extension to $\mathrm{cl}\mathbb{S}_\gamma[\Omega]^-$, which we denote by $w^-[\partial\Omega, S_{\gamma q,n}, \mu]$ (cf. *e.g.*, [30, §3].)

Similarly, $w[\partial\Omega, S_n, \mu]_{|\Omega}$ admits a continuous extension to $\mathrm{cl}\Omega$, which we denote by $w^+[\partial\Omega, S_n, \mu]$ and $w[\partial\Omega, S_n, \mu]_{|\Omega^-}$ admits a continuous extension to $\mathrm{cl}\Omega^-$, which we denote by $w^-[\partial\Omega, S_n, \mu]$ (cf. *e.g.*, Miranda [44], [35, Thm. 3.1].)

In the specific case in which H equals S_n, we omit S_n and we simply write $v[\partial\Omega, \mu]$, $w[\partial\Omega, \mu]$, $w_*[\partial\Omega, \mu]$ instead of $v[\partial\Omega, S_n, \mu]$, $w[\partial\Omega, S_n, \mu]$, $w_*[\partial\Omega, S_n, \mu]$, respectively.

3 Formulation of problem (4) in terms of integral equations

As a first step, we transform our problem so as to remove the parameters δ and γ from the domain of problem (4). We do so by exploiting the rule of change of variables.

We observe that a function $u \in C^{m,\alpha}(\mathrm{cl}\mathbb{S}(\epsilon, \delta, \gamma)^-)$ satisfies problem (4) if and only if the function

$$u(\delta\gamma\cdot) \in C^{m,\alpha}(\mathrm{cl}\mathbb{S}(\epsilon, 1, I)^-)\,,$$

satisfies the following auxiliary boundary value problem

$$\begin{cases} \sum_{j=1}^n \gamma_{jj}^{-2} \frac{\partial^2 u^\sharp}{\partial x_j^2}(x) = \delta^2 f(x) & \forall x \in \mathbb{S}(\epsilon, 1, I)^-\,, \\ u^\sharp \text{ is } q-\text{periodic in } \mathbb{S}(\epsilon, 1, I)^-\,, & \\ u^\sharp(x) = g(\epsilon^{-1}(x-p)) & \forall x \in \partial\Omega_{p,\epsilon}\,, \end{cases} \tag{7}$$

for all $(\epsilon, \delta, \gamma) \in]0, \epsilon_0[\times]0, +\infty[\times\mathbb{D}_n^+(\mathbb{R})$. In particular, the auxiliary problem (7) has a unique solution in $C_q^{m,\alpha}(\mathrm{cl}\mathbb{S}(\epsilon, 1, I)^-)$, which we denote by

$$u^\sharp(\epsilon, \delta, \gamma, \cdot)\,,$$

and

$$u(\epsilon, \delta, \gamma, x) = u^\sharp(\epsilon, \delta, \gamma, \delta^{-1}\gamma^{-1}x) \qquad \forall x \in \mathrm{cl}\mathbb{S}(\epsilon, \delta, \gamma)^-\,. \tag{8}$$

Next we convert problem (7) into a system of integral equations. To do so, we need the following statement. For a proof, we refer to [46, Prop. 2.9, p. 339].

Proposition 3.1. *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0, 1[$. Let $\gamma \in \mathbb{D}_n^+(\mathbb{R})$. Let $\mathbb{I}$ be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$ such that $\mathbb{R}^n \setminus \mathrm{cl}\mathbb{I}$ is connected and such that $\mathrm{cl}\mathbb{I} \subseteq \gamma Q$. Then the map $M[\cdot, \cdot]$ from $C^{m,\alpha}(\partial\mathbb{I})_0 \times \mathbb{R}$ to $C^{m,\alpha}(\partial\mathbb{I})$ defined by*

$$M[\mu, \xi](x) \equiv -\frac{1}{2}\mu(x) + w[\partial\mathbb{I}, S_{\gamma q,n}, \mu](x) + \xi \qquad \forall x \in \partial\mathbb{I}\,,$$

for all $(\mu, \xi) \in C^{m,\alpha}(\partial\mathbb{I})_0 \times \mathbb{R}$ is a linear homeomorphism from $C^{m,\alpha}(\partial\mathbb{I})_0 \times \mathbb{R}$ onto $C^{m,\alpha}(\partial\mathbb{I})$. Moreover, for each γq-periodic function u in $C^{m,\alpha}(\mathrm{cl}(\mathbb{S}_\gamma[\mathbb{I}]^-))$ such that $\Delta u = 0$ in $\mathbb{S}_\gamma[\mathbb{I}]^-$ there exists a unique pair $(\mu, \xi) \in C^{m,\alpha}(\partial\mathbb{I})_0 \times \mathbb{R}$ such that

$$u(x) = w^-[\partial\mathbb{I}, S_{\gamma q,n}, \mu](x) + \xi \qquad \forall x \in \mathrm{cl}\mathbb{S}_\gamma[\mathbb{I}]^-\,.$$

Such pair $(\mu, \xi) \in C^{m,\alpha}(\partial\mathbb{I})_0 \times \mathbb{R}$ is the unique solution of

$$M[\mu, \xi](x) = u(x) \qquad \forall x \in \partial\mathbb{I}\,.$$

Then we are in the position to prove the following.

Theorem 3.2. *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. Let $p \in Q$. Let Ω be as in (1). Let ϵ_0 be as in (2). Let $(\epsilon,\delta,\gamma) \in]0,\epsilon_0[\times]0,+\infty[\times\mathbb{D}_n^+(\mathbb{R})$. Let f be as in (3). Let $g \in C^{m,\alpha}(\partial\Omega)$. Then problem (7) has a unique solution $u^\sharp(\epsilon,\delta,\gamma,\cdot)$ in $C_q^{m,\alpha}(\mathrm{cl}\mathbb{S}(\epsilon,1,I)^-)$, which is delivered by the formula*

$$u^\sharp(\epsilon,\delta,\gamma,x) \equiv \omega^\sharp(\epsilon,\delta,\gamma,x) + \delta^2 P_{\gamma q,n}[\gamma Q, f(\gamma^{-1}\cdot)](\gamma x) \qquad \forall x \in \mathrm{cl}\mathbb{S}(\epsilon,1,I)^- \,, \tag{9}$$

where

$$\omega^\sharp(\epsilon,\delta,\gamma,x) \equiv w^-[\partial\gamma\Omega_{p,\epsilon}, S_{\gamma q,n}, \theta(\gamma^{-1}\epsilon^{-1}(\cdot-\gamma p))](\gamma x) + c \qquad \forall x \in \mathrm{cl}\mathbb{S}(\epsilon,1,I)^- \,, \tag{10}$$

and where $(\theta,c) \in C^{m,\alpha}(\partial\Omega)\times\mathbb{R}$ is the unique solution of the following system of integral equations

$$\begin{aligned} -\frac{1}{2}\theta(t) - \det\gamma \int_{\partial\Omega} (\gamma^{-1}\nu_\Omega(s)) DS_n(\gamma(t-s))\theta(s)\,d\sigma_s \\ -\epsilon^{n-1}\det\gamma \int_{\partial\Omega}(\gamma^{-1}\nu_\Omega(s)) DR_{\gamma q,n}(\epsilon\gamma(t-s))\theta(s)\,d\sigma_s + c \\ = g(t) - \delta^2 \det\gamma \int_Q S_{\gamma q,n}(\gamma(p+\epsilon t - s)) f(s)\,ds \qquad \forall t \in \partial\Omega\,, \end{aligned} \tag{11}$$

$$\int_{\partial\Omega} \theta\sigma[\gamma]\,d\sigma = 0\,, \tag{12}$$

where

$$\sigma[\gamma](s) \equiv \det\gamma|\gamma^{-1}\nu_\Omega(s)| \qquad \forall s \in \partial\Omega\,.$$

Proof. By the rule of change of variables, a function $u^\sharp \in C_q^{m,\alpha}(\mathrm{cl}\mathbb{S}(\epsilon,1,I)^-)$ solves problem (7) if and only if the function $u^\sharp(\gamma^{-1}\cdot)$ satisfies the following boundary value problem

$$\begin{cases} \Delta u(x) = \delta^2 f(\gamma^{-1}x) & \forall x \in \mathbb{S}(\epsilon,1,\gamma)^- \,, \\ u \text{ is } \gamma q - \text{periodic in } \mathbb{S}(\epsilon,1,\gamma)^- \,, & \\ u(x) = g(\gamma^{-1}\epsilon^{-1}(x-\gamma p)) & \forall x \in \partial\gamma\Omega_{p,\epsilon}\,, \end{cases} \tag{13}$$

a problem which is well known to have a unique solution in $C^{m,\alpha}(\mathrm{cl}\mathbb{S}(\epsilon,1,\gamma)^-)$. Moreover, u satisfies problem (13) if and only if the function

$$\omega(x) \equiv u(x) - \delta^2 P_{\gamma q,n}[\gamma Q, f(\gamma^{-1}\cdot)](x) \qquad \forall x \in \mathrm{cl}\mathbb{S}(\epsilon,1,\gamma)^- \,,$$

satisfies the following boundary value problem

$$\begin{cases} \Delta \omega(x) = 0 & \forall x \in \mathbb{S}(\epsilon,1,\gamma)^- \,, \\ \omega \text{ is } \gamma q - \text{periodic in } \mathbb{S}(\epsilon,1,\gamma)^- \,, & \\ \omega(x) = g(\gamma^{-1}\epsilon^{-1}(x-\gamma p)) - \delta^2 P_{\gamma q,n}[\gamma Q, f(\gamma^{-1}\cdot)](x) & \forall x \in \partial\gamma\Omega_{p,\epsilon}\,. \end{cases} \tag{14}$$

Thus if $u^\sharp \in C_q^{m,\alpha}(\mathrm{cl}\mathbb{S}(\epsilon,1,I)^-)$ solves problem (7), then Proposition 3.1 applied to problem (14) implies that there exists a unique pair $(\theta,c) \in C^{m,\alpha}(\partial\Omega)\times\mathbb{R}$ such that

$$\int_{\partial\gamma\Omega_{p,\epsilon}} \theta(\gamma^{-1}\epsilon^{-1}(y-\gamma p))\,d\sigma_y = 0\,, \tag{15}$$

and such that

$$u^\sharp(\gamma^{-1}x) = w^-[\partial\gamma\Omega_{p,\epsilon}, S_{\gamma q,n}, \theta(\gamma^{-1}\epsilon^{-1}(\cdot-\gamma p))](x) + c + \delta^2 P_{\gamma q,n}[\gamma Q, f(\gamma^{-1}\cdot)](x)\,, \tag{16}$$

for all $x \in \mathrm{cl}\mathbb{S}(\epsilon,1,\gamma)^-$, and that (θ,c) is the unique pair of $C^{m,\alpha}(\partial\Omega)\times\mathbb{R}$ such that

$$\begin{aligned} -\frac{1}{2}\theta(\gamma^{-1}\epsilon^{-1}(x-\gamma p)) + w[\partial\gamma\Omega_{p,\epsilon}, S_{\gamma q,n}, \theta(\gamma^{-1}\epsilon^{-1}(\cdot-\gamma p))](x) + c \\ = g(\gamma^{-1}\epsilon^{-1}(x-\gamma p)) - \delta^2 P_{\gamma q,n}[\gamma Q, f(\gamma^{-1}\cdot)](x) \qquad \forall x \in \partial\gamma\Omega_{p,\epsilon}\,, \end{aligned} \tag{17}$$

and such that (15) holds. Equation (17) can be rewritten as

$$\begin{aligned} -\frac{1}{2}\theta(t) - \epsilon^{n-1}\int_{\partial\Omega} \nu_{\gamma\Omega_{p,\epsilon}}(\gamma p + \epsilon\gamma s) DS_{\gamma q,n}(\gamma p + \epsilon\gamma t - (\gamma p + \epsilon\gamma s))\theta(s)\sigma[\gamma](s)\,d\sigma_s + c \\ = g(t) - \delta^2 \int_Q S_{\gamma q,n}(\gamma p + \epsilon\gamma t - \gamma s) f(s)\det\gamma\,ds \qquad \forall t \in \partial\Omega\,. \end{aligned} \tag{18}$$

By rewriting once more equation (18), we deduce that (15), (16) can be rewritten as

$$
\begin{aligned}
-\frac{1}{2}\theta(t) - \epsilon^{n-1}\det\gamma \int_{\partial\Omega} (\gamma^{-1}\nu_\Omega(s)) DS_{\gamma q,n}(\epsilon\gamma(t-s))\theta(s)\,d\sigma_s + c \\
= g(t) - \delta^2 \det\gamma \int_Q S_{\gamma q,n}(\gamma(p+\epsilon t - s)) f(s)\,ds \qquad \forall t \in \partial\Omega\,,
\end{aligned}
$$

$$
\int_{\partial\Omega} \theta\sigma[\gamma]\,d\sigma = 0\,,
$$

and thus in the form of (11). Indeed, $\nu_{\gamma\Omega_{p,\epsilon}}(\gamma p + \epsilon\gamma s) = \frac{\gamma^{-1}\nu_\Omega(s)}{|\gamma^{-1}\nu_\Omega(s)|}$ for all $s \in \partial\Omega$.

Conversely, by reading backward the above arguments, we can show that that if (θ, c) solves (11), (12), then the function $u^\sharp(\epsilon,\delta,\gamma,\cdot)$ delivered by (9) solves (7).

On the other hand, if (θ_1, c_1), $(\theta_2, c_2) \in C^{m,\alpha}(\partial\Omega)\times\mathbb{R}$ solve equations (11), (12), then

$$
w^-[\partial\gamma\Omega_{p,\epsilon}, S_{\gamma q,n}, \theta_1(\gamma^{-1}\epsilon^{-1}(\cdot - \gamma p))](x) + c_1 = w^-[\partial\gamma\Omega_{p,\epsilon}, S_{\gamma q,n}, \theta_2(\gamma^{-1}\epsilon^{-1}(\cdot - \gamma p))](x) + c_2
$$

for all $x \in \partial\gamma\Omega_{p,\epsilon}$. Then by uniqueness of Proposition 3.1, we deduce that $\theta_1 = \theta_2$, $c_1 = c_2$. □

Next we note that if $(\theta, c) \in C^{m,\alpha}(\partial\Omega)\times\mathbb{R}$ and if we let (ϵ,δ,γ) tend to $(0,0,\tilde\gamma)$ in equations (11), (12), we obtain a system which we address to as the 'limiting system', and which has the following form

$$
-\frac{1}{2}\theta(t) - \det\tilde\gamma \int_{\partial\Omega} (\tilde\gamma^{-1}\nu_\Omega(s)) DS_n(\tilde\gamma(t-s))\theta(s)\,d\sigma_s + c = g(t) \qquad \forall t\in\partial\Omega\,, \tag{19}
$$

$$
\int_{\partial\Omega} \theta\sigma[\tilde\gamma]\,d\sigma = 0\,. \tag{20}
$$

Then we have the following Theorem, which shows the unique solvability of system (19), (20) and its link with a boundary value problem, which we shall address to as the 'limiting boundary value problem'. To do so, we proceed as in the proof of [46, Lem. 3.4].

Theorem 3.3. *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$. Let $p \in Q$. Let Ω be as in (1). Let $\tilde\gamma \in \mathbb{D}_n^+(\mathbb{R})$. Let $g \in C^{m,\alpha}(\partial\Omega)$, $d \in \mathbb{R}$. Let $\tilde\tau$ be the unique solution in $C^{m-1,\alpha}(\partial\Omega)$ of the following problem*

$$
\begin{cases}
-\frac{1}{2}\tau(t) + \det\tilde\gamma \int_{\partial\Omega} (\tilde\gamma^{-1}\nu_\Omega(t)) DS_n(\tilde\gamma(t-s))\tau(s)\,d\sigma_s = 0 \quad \forall t\in\partial\Omega\,, \\
\int_{\partial\Omega} \tau\sigma[\tilde\gamma]\,d\sigma = 1\,.
\end{cases} \tag{21}
$$

Then the following statements hold.

(i) There exists a unique pair $(\theta, c) \in C^{m,\alpha}(\partial\Omega)\times\mathbb{R}$ which satisfies the system of equation (19) and condition

$$
\int_{\partial\Omega} \theta\sigma[\tilde\gamma]\,d\sigma = d\,. \tag{22}
$$

In particular, the limiting system (19), (20) has one and only one solution $(\tilde\theta, \tilde c)$ in the space $C^{m,\alpha}(\partial\Omega)\times\mathbb{R}$. Moreover,

$$
\tilde c = \int_{\partial\Omega} g\tilde\tau\sigma[\tilde\gamma]\,d\sigma\,. \tag{23}
$$

(ii) The 'limiting boundary value problem'

$$
\begin{cases}
\sum_{j=1}^n \tilde\gamma_{jj}^{-2} \frac{\partial^2 u}{\partial x_j^2}(x) = 0 & \forall x \in \mathbb{R}^n\setminus \mathrm{cl}\Omega\,, \\
u(x) = g(x) & \forall x \in \partial\Omega\,, \\
\lim_{x\to\infty} u(x) = \tilde c\,,
\end{cases} \tag{24}
$$

has one and only one solution $\tilde u$ in $C^{m,\alpha}_{\mathrm{loc}}(\mathbb{R}^n\setminus\Omega)$. Moreover,

$$
\tilde u(x) = w[\partial\tilde\gamma\Omega, \tilde\theta(\tilde\gamma^{-1}\cdot)](\tilde\gamma x) + \tilde c \qquad \forall x \in \mathbb{R}^n\setminus\Omega\,. \tag{25}
$$

Proof. We first consider statement (i). By a change of variables, we can rewrite problem (21) in the form

$$
\begin{cases}
-\frac{1}{2}\tau(\tilde\gamma^{-1}t) + w_*[\partial\tilde\gamma\Omega, \tau(\tilde\gamma^{-1}\cdot)](t) = 0 \quad \forall t\in\partial\tilde\gamma\Omega\,, \\
\int_{\partial\tilde\gamma\Omega} \tau(\tilde\gamma^{-1}s)\,d\sigma_s = 1\,.
\end{cases} \tag{26}
$$

By classical potential theory, problem (26) has a unique solution $\tilde{\tau} \in C^0(\partial\Omega)$ (cf. *e.g.*, Folland [21, Ch. 3].) Then by Schauder regularity theory, the function $\tilde{\tau}(\tilde{\gamma}^{-1}\cdot)$ belongs to $C^{m-1,\alpha}(\partial\Omega)$ (cf. *e.g.*, [28, App. A].) Moreover, the adjoint equation

$$-\frac{1}{2}\tilde{\theta}(\tilde{\gamma}^{-1}t) + w[\partial\tilde{\gamma}\Omega, \tilde{\theta}(\tilde{\gamma}^{-1}\cdot)](t) = g(\tilde{\gamma}^{-1}t) - c \qquad \forall t \in \partial\tilde{\gamma}\Omega$$

can have a solution $\tilde{\theta} \in C^{m,\alpha}(\partial\Omega)$ if and only if

$$\int_{\partial\Omega} (g-c)\tilde{\tau}\sigma[\tilde{\gamma}]\,d\sigma = 0\,,$$

and thus if and only if condition (23) holds. Since the kernel of the operator $-\frac{1}{2}I + w[\partial\tilde{\gamma}\Omega, \cdot]_{|\partial\Omega}$ is well known to be the set of constant functions, condition (22) guarantees the uniqueness of $\tilde{\theta}$ (cf. *e.g.*, Folland [21, Ch. 3].)

Next we consider statement (ii). By the rule of change of variables, the function $\tilde{u}$ defined by (25) satisfies the first and the third equation of problem (24). By classical jump properties of double layer potentials and by equation (19), the function $\tilde{u}$ satisfies also the second condition in (24). On the other hand, up to a change of variables, problem (24) is an exterior Dirichlet problem for the Laplace operator and is well known to have up to one classical solution, which must necessarily coincide with $\tilde{u}$. □

We are now ready to analyze equations (11), (12) around the degenerate case in which $(\epsilon, \delta, \gamma) = (0, 0, \tilde{\gamma})$. Thus we introduce the following.

Theorem 3.4. *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$. Let $p \in Q$. Let Ω be as in (1). Let ϵ_0 be as in (2). Let $\tilde{\gamma} \in \mathbb{D}^+_n(\mathbb{R}^n)$. Let f be as in (3). Let $g \in C^{m,\alpha}(\partial\Omega)$.*

Let $\Lambda \equiv (\Lambda_j)_{j=1,2}$ be the map from $]-\epsilon_0, \epsilon_0[\times\mathbb{R}\times\mathbb{D}^+_n(\mathbb{R})\times C^{m,\alpha}(\partial\Omega)\times\mathbb{R}$ to $C^{m,\alpha}(\partial\Omega)\times\mathbb{R}$ defined by

$$\begin{aligned}
\Lambda_1[\epsilon,\delta,\gamma,\theta,c](t) \equiv -\frac{1}{2}\theta(t) - \det\gamma \int_{\partial\Omega} (\gamma^{-1}\nu_\Omega(s)) DS_n(\gamma(t-s))\theta(s)\,d\sigma_s \\
-\epsilon^{n-1}\det\gamma\int_{\partial\Omega}(\gamma^{-1}\nu_\Omega(s))DR_{\gamma q,n}(\epsilon\gamma(t-s))\theta(s)\,d\sigma_s + c \\
-g(t) + \delta^2 \det\gamma \int_Q S_{\gamma q,n}(\gamma(p+\epsilon t - s))f(s)\,ds \qquad \forall t\in\partial\Omega\,,
\end{aligned}$$

$$\Lambda_2[\epsilon,\delta,\gamma,\theta,c] \equiv \int_{\partial\Omega}\theta\sigma[\gamma]\,d\sigma\,,$$

for all $(\epsilon,\delta,\gamma,\theta,c) \in]-\epsilon_0,\epsilon_0[\times\mathbb{R}\times\mathbb{D}^+_n(\mathbb{R})\times C^{m,\alpha}(\partial\Omega)\times\mathbb{R}$. Then the following statements hold.

(i) Equation $\Lambda[0,0,\tilde{\gamma},\theta,c] = 0$ is equivalent to the limiting system (19), (20) and has one and only one solution $(\tilde{\theta},\tilde{c}) \in C^{m,\alpha}(\partial\Omega)\times\mathbb{R}$ (cf. Theorem 3.3.)

(ii) If $(\epsilon,\delta,\gamma) \in]0,\epsilon_0[\times]0,+\infty[\times\mathbb{D}^+_n(\mathbb{R})$, then the equation $\Lambda[\epsilon,\delta,\gamma,\theta,c] = 0$ is equivalent to the system (11), (12), which has one and only one solution $(\theta[\epsilon,\delta,\gamma], c[\epsilon,\delta,\gamma])$ in $C^{m,\alpha}(\partial\Omega)\times\mathbb{R}$.

(iii) There exist $(\epsilon',\delta') \in]0,\epsilon_0[\times]0,+\infty[$ and an open neighborhood $\tilde{\Gamma}$ of $\tilde{\gamma}$ in $\mathbb{D}^+_n(\mathbb{R})$ and an open neighborhood $\mathcal{U}$ of $(\tilde{\theta},\tilde{c})$ in $C^{m,\alpha}(\partial\Omega)\times\mathbb{R}$ and a real analytic map $(\Theta[\cdot,\cdot,\cdot], C[\cdot,\cdot,\cdot])$ from $]-\epsilon',\epsilon'[\times]-\delta',\delta'[\times\tilde{\Gamma}$ to $\mathcal{U}$ such that the set of zeros of the map Λ in $]-\epsilon',\epsilon'[\times]-\delta',\delta'[\times\tilde{\Gamma}\times\mathcal{U}$ coincides with the graph of $(\Theta[\cdot,\cdot,\cdot], C[\cdot,\cdot,\cdot])$. In particular,

$$\begin{aligned}
&(\Theta[0,0,\tilde{\gamma}], C[0,0,\tilde{\gamma}]) = (\tilde{\theta},\tilde{c})\,,\\
&(\Theta[\epsilon,\delta,\gamma], C[\epsilon,\delta,\gamma]) = (\theta[\epsilon,\delta,\gamma], c[\epsilon,\delta,\gamma]) \qquad \forall(\epsilon,\delta,\gamma)\in]0,\epsilon'[\times]0,\delta'[\times\tilde{\Gamma}\,.
\end{aligned}$$

Proof. Statements (i) and (ii) are an immediate consequence of Theorems 3.2 and 3.3 and of the definition of Λ.

We now turn to show that Λ is real analytic in a neighborhood of $(0,0,\tilde{\gamma},\tilde{\theta},\tilde{c})$. The analyticity of Λ_2 in the domain of Λ follows by the analyticity of the pointwise product in $C^{m-1,\alpha}(\partial\Omega)$ and by the analyticity of $\sigma[\cdot]$ from $\mathbb{D}^+_n(\mathbb{R})$ to $C^{m-1,\alpha}(\partial\Omega)$. We now turn to consider Λ_1. By [36, Thm. 4.11 (iii)], the map from $\mathbb{D}^+_n(\mathbb{R})\times C^{m,\alpha}(\partial\Omega)$ to $C^{m,\alpha}(\partial\Omega)$ which takes (γ,θ) to

$$\det\gamma\int_{\partial\Omega}(\gamma^{-1}\nu_\Omega(s))DS_n(\gamma(t-s))\theta(s)\,d\sigma_s = w[\partial\gamma\Omega,\theta(\gamma^{-1}\cdot)](\gamma t) \qquad \forall t\in\partial\Omega\,, \tag{27}$$

is real analytic.

Next we turn to consider the second integral operator in the definition of Λ_1. Let Ω' be a bounded open subset of $\mathbb{R}^n$ of class C^1 such that $\mathrm{cl}\Omega' \subseteq \mathbb{R}^n \setminus q(\mathbb{Z}^n \setminus \{0\})$, $0 \in \Omega'$. Since $(\partial\Omega) - (\partial\Omega)$ is compact, there exists $\epsilon' \in]0, \epsilon_0[$ such that

$$\epsilon(t-s) \subseteq \Omega' \qquad \forall \epsilon \in]-\epsilon', \epsilon'[\,, \qquad \forall s, t \in \partial\Omega\,.$$

Then by Theorem A.2 (i) of the Appendix, the map $DR_{\gamma q,n}(\epsilon\gamma(t-s))$ is analytic in the variable (ϵ, γ, t, s) in an open neighborhood of $]-\epsilon', \epsilon'[\times\tilde{\Gamma} \times (\partial\Omega)^2$. Then by a result on integral operators with real analytic kernels and with no singularity (cf. [31, Prop. 4.1 (i)]), the map from $]-\epsilon', \epsilon'[\times\tilde{\Gamma} \times C^{m,\alpha}(\partial\Omega)$ to $C^{m,\alpha}(\partial\Omega)$ which takes $(\epsilon, \gamma, \theta)$ to the function

$$\int_{\partial\Omega} (\gamma^{-1}\nu_\Omega(s)) DR_{\gamma q,n}(\epsilon\gamma(t-s))\theta(s)\, d\sigma_s \qquad \forall t \in \partial\Omega\,, \tag{28}$$

is analytic. We now consider the third integral operator in the definition of Λ_1. Let $\rho \in]0, +\infty[$ be as in (6). Then possibly shrinking $\tilde{\Gamma}$, Propositions A.4 and A.5 and Lemma A.7 of the Appendix imply that the map from $]-\epsilon', \epsilon'[\times\tilde{\Gamma}$ to $C^{m,\alpha}(\partial\Omega)$ which takes (ϵ, γ) to the function

$$\int_Q S_{\gamma q,n}(\gamma(p+\epsilon t - s)) f(s)\, ds \qquad \forall t \in \partial\Omega\,, \tag{29}$$

is real analytic. By the analytic dependence of the maps in (27)–(29), we deduce that Λ_1 is real analytic in $]-\epsilon', \epsilon'[\times\mathbb{R} \times \tilde{\Gamma} \times C^{m,\alpha}(\partial\Omega) \times \mathbb{R}$.

Next we turn to prove that the differential $\partial_{(\theta,c)}\Lambda[0,0,\tilde{\gamma},\tilde{\theta},\tilde{c}]$ of Λ at $(0,0,\tilde{\gamma},\tilde{\theta},\tilde{c})$ with respect to the variable (θ, c) is a linear homeomorphism from $C^{m,\alpha}(\partial\Omega) \times \mathbb{R}$ onto itself. By standard calculus in Banach spaces, the differential of Λ at $(0,0,\tilde{\gamma},\tilde{\theta},\tilde{c})$ with respect to the variable (θ, c) is delivered by the following formula

$$\begin{aligned}
&\partial_{(\theta,c)}\Lambda_1[0,0,\tilde{\gamma},\tilde{\theta},\tilde{c}](\overline{\theta},\overline{c})(t) \\
&\qquad\qquad = -\frac{1}{2}\overline{\theta}(t) - \det\tilde{\gamma}\int_{\partial\Omega}(\tilde{\gamma}^{-1}\nu_\Omega(s))DS_n(\tilde{\gamma}(t-s))\overline{\theta}(s)\, d\sigma_s + \overline{c} \qquad \forall t \in \partial\Omega\,, \\
&\partial_{(\theta,c)}\Lambda_2[0,0,\tilde{\gamma},\tilde{\theta},\tilde{c}](\overline{\theta},\overline{c}) = \int_{\partial\Omega}\overline{\theta}\sigma[\tilde{\gamma}]\, d\sigma\,,
\end{aligned}$$

for all $(\overline{\theta},\overline{c}) \in C^{m,\alpha}(\partial\Omega) \times \mathbb{R}$. We now show that $\partial_{(\theta,c)}\Lambda[0,0,\tilde{\gamma},\tilde{\theta},\tilde{c}]$ is a bijection. To do so, we show that if $(h, d) \in C^{m,\alpha}(\partial\Omega) \times \mathbb{R}$, then the equation

$$\partial_{(\theta,c)}\Lambda[0,0,\tilde{\gamma},\tilde{\theta},\tilde{c}](\overline{\theta},\overline{c}) = (h, d)\,, \tag{30}$$

has a unique solution $(\overline{\theta},\overline{c}) \in C^{m,\alpha}(\partial\Omega) \times \mathbb{R}$. By changing the variable with the map $\tilde{\gamma}$ in the first component of (30), we rewrite (30) in the following way

$$\begin{cases} -\frac{1}{2}\overline{\theta}(t) - \det\tilde{\gamma}\int_{\partial\Omega}(\tilde{\gamma}^{-1}\nu_\Omega(s))DS_n(\tilde{\gamma}(t-s))\overline{\theta}(s)\, d\sigma_s + \overline{c} = h(t) & \forall t \in \partial\Omega\,, \\ \int_{\partial\tilde{\gamma}\Omega}\overline{\theta}(\tilde{\gamma}^{-1}y)\, d\sigma_y = d\,. \end{cases} \tag{31}$$

By Theorem 3.3, problem (31) has a unique solution $(\overline{\theta},\overline{c})$ in $C^{m,\alpha}(\partial\Omega) \times \mathbb{R}$. Hence, $\partial_{(\theta,c)}\Lambda[0,0,\tilde{\gamma},\tilde{\theta},\tilde{c}]$ is a bijection. Then by the Open Mapping Theorem, the operator $\partial_{(\theta,c)}\Lambda[0,0,\tilde{\gamma},\tilde{\theta},\tilde{c}]$ is a homeomorphism. Since Λ is analytic, statement (iii) is an immediate consequence of statements (i), (ii) and of the Implicit Function Theorem in Banach spaces (cf. *e.g.*, Deimling [19, Thm. 15.3].) □

4 A functional analytic representation theorem for the solution of the auxiliary problem (7)

We now prove a representation theorem for the solution $u^\sharp(\epsilon, \delta, \gamma, \cdot)$ of problem (7) (cf. (9).)

Proposition 4.1. *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. Let $p \in Q$. Let Ω be as in (1). Let ϵ_0 be as in (2). Let $\tilde{\gamma} \in \mathbb{D}_n^+(\mathbb{R})$. Let f be as in (3). Let $g \in C^{m,\alpha}(\partial\Omega)$.*

Let $\tilde{u}$, $\tilde{c}$ be as in Theorem 3.3. Let ϵ', δ', $\tilde{\Gamma}$ be as in Theorem 3.4 (iii). Then the following statements hold.

(i) There exists $\rho' \in]0, +\infty[$ such that the map $\mathcal{P}$ from $\tilde{\Gamma}$ to $C^0_{q,\omega,\rho'}(\mathbb{R}^n)$ defined by

$$\mathcal{P}[\gamma](x) \equiv P_{\gamma q,n}[\gamma Q, f(\gamma^{-1}\cdot)](\gamma x) \qquad \forall x \in \mathbb{R}^n\,,$$

for all $\gamma \in \tilde{\Gamma}$, is real analytic.

(ii) Let $\tilde{\Omega}$ be an open subset of $\mathbb{R}^n$ with nonzero distance from $p+q\mathbb{Z}^n$. Then there exist $\epsilon^*_{\tilde{\Omega}} \in]0,\epsilon'[$ such that

$$\mathrm{cl}\tilde{\Omega} \subseteq \mathbb{S}[\Omega_{p,\epsilon}]^- \qquad \forall \epsilon \in [-\epsilon^*_{\tilde{\Omega}}, \epsilon^*_{\tilde{\Omega}}] \,,$$

and $\epsilon_{\tilde{\Omega}} \in]0,\epsilon^*_{\tilde{\Omega}}[$ such that $\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde{\Omega}}}]^- \subseteq \mathbb{S}[\Omega_{p,\epsilon}]^-$ for all $\epsilon \in [-\epsilon_{\tilde{\Omega}}, \epsilon_{\tilde{\Omega}}]$, and a real analytic map $V^\sharp_{\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde{\Omega}}}]^-}$ from $]-\epsilon_{\tilde{\Omega}}, \epsilon_{\tilde{\Omega}}[\times]-\delta',\delta'[\times\tilde{\Gamma}$ to $C^{m,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde{\Omega}}}]^-)$ such that

$$\omega^\sharp(\epsilon,\delta,\gamma,x) = V^\sharp_{\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde{\Omega}}}]^-}[\epsilon,\delta,\gamma](x) \qquad \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde{\Omega}}}]^- \,,$$
$$u^\sharp(\epsilon,\delta,\gamma,x) = V^\sharp_{\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde{\Omega}}}]^-}[\epsilon,\delta,\gamma](x) + \delta^2\mathcal{P}[\gamma](x) \qquad \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde{\Omega}}}]^- \,,$$

for all $(\epsilon,\delta,\gamma) \in]0,\epsilon_{\tilde{\Omega}}[\times]0,\delta'[\times\tilde{\Gamma}$. Moreover,

$$V^\sharp_{\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde{\Omega}}}]^-}[0,0,\tilde{\gamma}](x) = \tilde{c} \,, \tag{32}$$

for all $x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde{\Omega}}}]^-$.

(iii) Let $\tilde{\Omega}$ be a bounded open subset of $\mathbb{R}^n \setminus \mathrm{cl}\Omega$. Then there exist $\epsilon_{\tilde{\Omega},r} \in]0,\epsilon'[$ and a real analytic map $V^{\sharp r}_{\tilde{\Omega}}$ from $]-\epsilon_{\tilde{\Omega},r}, \epsilon_{\tilde{\Omega},r}[\times]-\delta',\delta'[\times\tilde{\Gamma}$ to $C^{m,\alpha}(\mathrm{cl}\tilde{\Omega})$ and a real analytic map $\mathcal{P}^r_{\tilde{\Omega}}$ from $]-\epsilon_0,\epsilon_0[\times\tilde{\Gamma}$ to $C^{m,\alpha}(\mathrm{cl}\tilde{\Omega})$ such that

$$p + \epsilon\mathrm{cl}\tilde{\Omega} \subseteq \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^- \qquad \forall \epsilon \in]-\epsilon_{\tilde{\Omega},r}, \epsilon_{\tilde{\Omega},r}[\,,$$

and

$$\omega^\sharp(\epsilon,\delta,\gamma,p+\epsilon t) = V^{\sharp r}_{\tilde{\Omega}}[\epsilon,\delta,\gamma](t) \qquad \forall t \in \mathrm{cl}\tilde{\Omega} \,,$$
$$u^\sharp(\epsilon,\delta,\gamma,p+\epsilon t) = V^{\sharp r}_{\tilde{\Omega}}[\epsilon,\delta,\gamma](t) + \delta^2\mathcal{P}^r_{\tilde{\Omega}}[\epsilon,\gamma](t) \qquad \forall t \in \mathrm{cl}\tilde{\Omega} \,,$$

for all $(\epsilon,\delta,\gamma) \in]0,\epsilon_{\tilde{\Omega},r}[\times]0,\delta'[\times\tilde{\Gamma}$. Moreover,

$$V^{\sharp r}_{\tilde{\Omega}}[0,0,\tilde{\gamma}](t) = \tilde{u}(t) \qquad \forall t \in \mathrm{cl}\tilde{\Omega} \,. \tag{33}$$

(iv) There exists a real analytic map $J^\sharp$ from $]-\epsilon',\epsilon'[\times]-\delta',\delta'[\times\tilde{\Gamma}$ to $\mathbb{R}$ such that

$$\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}} u^\sharp(\epsilon,\delta,\gamma,x)\,dx = J^\sharp[\epsilon,\delta,\gamma] + \delta^2\int_Q \mathcal{P}[\gamma]\,dx \,,$$

for all $(\epsilon,\delta,\gamma) \in]0,\epsilon'[\times]0,\delta'[\times\tilde{\Gamma}$. Moreover,

$$J^\sharp[0,0,\tilde{\gamma}] = \tilde{c}m_n(Q) \,. \tag{34}$$

Proof. (i) Since

$$P_{\gamma q,n}[\gamma Q, f(\gamma^{-1}\cdot)](\gamma x) = \int_Q S_{\gamma q,n}(\gamma x - \gamma s)f(s)\det\gamma\,ds \qquad \forall x \in \mathbb{R}^n \,,$$

for all $\gamma \in \tilde{\Gamma}$, the existence of ρ' follows by Proposition A.4 and Lemma A.7 of the Appendix.

Next we turn to statement (ii). Let $\epsilon^*_{\tilde{\Omega}}$, $\epsilon_{\tilde{\Omega}}$ be as in Lemma A.9 (i) of the Appendix. By definition of $\omega^\sharp(\epsilon,\delta,\gamma,\cdot)$, and by Theorem 3.2, and by Theorem 3.4 (iii), we have

$$\begin{aligned}\omega^\sharp(\epsilon,\delta,\gamma,x) &= w[\partial\gamma\Omega_{p,\epsilon}, S_{\gamma q,n}, \Theta[\epsilon,\delta,\gamma](\gamma^{-1}\epsilon^{-1}(\cdot-\gamma p))](\gamma x) + C[\epsilon,\delta,\gamma] \\ &= -\epsilon^{n-1}\det\gamma\int_{\partial\Omega}(\gamma^{-1}\nu_\Omega(s))DS_{\gamma q,n}(\gamma(x-p-\epsilon s))\Theta[\epsilon,\delta,\gamma](s)\,d\sigma_s + C[\epsilon,\delta,\gamma] \\ u^\sharp(\epsilon,\delta,\gamma,x) &= \omega^\sharp(\epsilon,\delta,\gamma,x) + \delta^2\int_Q S_{\gamma q,n}(\gamma x - \gamma s)f(s)\det\gamma\,ds\end{aligned}$$

for all $x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde{\Omega}}}]^-$ and for all $(\epsilon,\delta,\gamma) \in]0,\epsilon_{\tilde{\Omega}}[\times]0,\delta'[\times\tilde{\Gamma}$.

Thus it is natural to set

$$V^\sharp_{\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde{\Omega}}}]^-}[\epsilon,\delta,\gamma](x)$$

$$\equiv -\epsilon^{n-1}\det\gamma\int_{\partial\Omega}(\gamma^{-1}\nu_\Omega(s))DS_{\gamma q,n}(\gamma(x-p-\epsilon s))\Theta[\epsilon,\delta,\gamma](s)\,d\sigma_s + C[\epsilon,\delta,\gamma]$$

for all $x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde\Omega}}]^-$ and for all $(\epsilon,\delta,\gamma)\in]-\epsilon_{\tilde\Omega},\epsilon_{\tilde\Omega}[\times]-\delta',\delta'[\times\tilde\Gamma$. Now it suffices to show that the right hand side of the above definition defines a real analytic map from $]-\epsilon_{\tilde\Omega},\epsilon_{\tilde\Omega}[\times]-\delta',\delta'[\times\tilde\Gamma$ to $C^{m,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde\Omega}}]^-)$. Let V be an open bounded connected subset of $\mathbb{R}^n$ such that

$$\mathrm{cl}Q\subseteq V,\qquad \mathrm{cl}V\cap(qz+\mathrm{cl}\Omega_{p,\epsilon^*_{\tilde\Omega}})=\emptyset\qquad \forall z\in\mathbb{Z}^n\setminus\{0\}\,,$$

Let $W\equiv V\setminus \mathrm{cl}\Omega_{p,\epsilon^*_{\tilde\Omega}}$. By (5), the function from $]-\epsilon_{\tilde\Omega},\epsilon_{\tilde\Omega}[\times\tilde\Gamma\times\mathrm{cl}W\times\partial\Omega$ to $\mathbb{R}$ which takes (ϵ,γ,x,s) to $DS_{\gamma q,n}(\gamma(x-p-\epsilon s))$ is real analytic. Then by a result on integral operators with real analytic kernels and with no singularity (cf. [31, Prop. 4.1 (i)]), and by the analyticity of Θ, we conclude that the function from $]-\epsilon_{\tilde\Omega},\epsilon_{\tilde\Omega}[\times]-\delta',\delta'[\times\tilde\Gamma$ to $C^{m,\alpha}(\mathrm{cl}W)$ which takes (ϵ,δ,γ) to the function

$$\int_{\partial\Omega}(\gamma^{-1}\nu_\Omega(s))DS_{\gamma q,n}(\gamma(x-p-\epsilon s))\Theta[\epsilon,\delta,\gamma](s)\,d\sigma_s\qquad\forall x\in\mathrm{cl}W\,,$$

is real analytic. Since the restriction operator from $\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde\Omega}}]^-$ to $\mathrm{cl}W$ induces an isomorphism from the space $C^{m,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde\Omega}}]^-)$ onto the subspace of $C^{m,\alpha}(\mathrm{cl}W)$ of the restrictions to $\mathrm{cl}W$ of q-periodic functions on $\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde\Omega}}]^-$, we conclude that the function from $]-\epsilon_{\tilde\Omega},\epsilon_{\tilde\Omega}[\times]-\delta',\delta'[\times\tilde\Gamma$ to $C^{m,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde\Omega}}]^-)$ which takes (ϵ,δ,γ) to the above function is analytic. Since $C[\cdot,\cdot,\cdot]$ is analytic, then also $V^\sharp_{\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde\Omega}}]^-}[\cdot,\cdot,\cdot]$ is analytic. Also, equality (32) follows by the definition of $V^\sharp_{\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde\Omega}}]^-}$ and by Theorem 3.4 (iii).

We now turn to prove statement (iii). By assumption, there exists $R>0$ such that $\mathrm{cl}\tilde\Omega\subseteq\mathbb{B}_n(0,R)$. Then we set $\Omega^*\equiv\mathbb{B}_n(0,R)\setminus\mathrm{cl}\Omega$. Let $\epsilon_{\Omega^*,r}$ be as in Lemma A.9 (ii) of the Appendix with $\epsilon_1=\epsilon'$. Then we take $\epsilon_{\tilde\Omega,r}\equiv\epsilon_{\Omega^*,r}$. It clearly suffices to show that $V^{\sharp r}_{\Omega^*}$ and $\mathcal{P}^r_{\Omega^*}$ exist and then to set $V^{\sharp r}_{\tilde\Omega}$ and $\mathcal{P}^r_{\tilde\Omega}$ equal to the composition of the restriction of $C^{m,\alpha}(\mathrm{cl}\Omega^*)$ to $C^{m,\alpha}(\mathrm{cl}\tilde\Omega)$ with $V^{\sharp r}_{\Omega^*}$ and $\mathcal{P}^r_{\Omega^*}$, respectively. By definition of $\omega^\sharp$ and by Theorems 3.2, 3.4 (iii), we have

$$\begin{aligned}
\omega^\sharp(\epsilon,\delta,\gamma,p+\epsilon t) &= w[\partial\gamma\Omega_{p,\epsilon},S_{\gamma q,n},\Theta[\epsilon,\delta,\gamma](\gamma^{-1}\epsilon^{-1}(\cdot-\gamma p))](\gamma(p+\epsilon t))+C[\epsilon,\delta,\gamma]\\
&= w^-[\partial\gamma\Omega,\Theta[\epsilon,\delta,\gamma](\gamma^{-1}\cdot)](\gamma t)\\
&\quad -\epsilon^{n-1}\det\gamma\int_{\partial\Omega}(\gamma^{-1}\nu_\Omega(s))DR_{\gamma q,n}(\epsilon\gamma(t-s))\Theta[\epsilon,\delta,\gamma](s)\,d\sigma_s+C[\epsilon,\delta,\gamma]\\
u^\sharp(\epsilon,\delta,\gamma,p+\epsilon t) &= \omega^\sharp(\epsilon,\delta,\gamma,p+\epsilon t)+\delta^2\int_Q S_{\gamma q,n}(\gamma(p+\epsilon t)-\gamma s)f(s)\det\gamma\,ds
\end{aligned}$$

for all $t\in\mathrm{cl}\Omega^*$, and for all $(\epsilon,\delta,\gamma)\in]0,\epsilon_{\tilde\Omega,r}[\times]0,\delta'[\times\tilde\Gamma$. Thus it is natural to set

$$\begin{aligned}
V^{\sharp r}_{\Omega^*}[\epsilon,\delta,\gamma](t) &\equiv w^-[\partial\gamma\Omega,\Theta[\epsilon,\delta,\gamma](\gamma^{-1}\cdot)](\gamma t)\\
&\quad -\epsilon^{n-1}\det\gamma\int_{\partial\Omega}(\gamma^{-1}\nu_\Omega(s))DR_{\gamma q,n}(\epsilon\gamma(t-s))\Theta[\epsilon,\delta,\gamma](s)\,d\sigma_s+C[\epsilon,\delta,\gamma]
\end{aligned}\tag{35}$$

$$\mathcal{P}^r_{\Omega^*}[\epsilon,\gamma](t)\equiv\int_Q S_{\gamma q,n}(\gamma(p+\epsilon t)-\gamma s)f(s)\det\gamma\,ds\,,\tag{36}$$

for all $t\in\mathrm{cl}\Omega^*$, and for all $(\epsilon,\delta,\gamma)\in]-\epsilon_{\tilde\Omega,r},\epsilon_{\tilde\Omega,r}[\times]-\delta',\delta'[\times\tilde\Gamma$. Now it suffices to prove that the right hand side of (35), (36) define real analytic maps from $]-\epsilon_{\tilde\Omega,r},\epsilon_{\tilde\Omega,r}[\times]-\delta',\delta'[\times\tilde\Gamma$ to $C^{m,\alpha}(\mathrm{cl}\Omega^*)$. By [36, Thm. 4.11 (iii)], the map from $\mathbb{D}^+_n(\mathbb{R})\times C^{m,\alpha}(\partial\Omega)$ to $C^{m,\alpha}(\mathrm{cl}\Omega^*)$ which takes (γ,θ) to $w^-[\partial\gamma\Omega,\theta(\gamma^{-1}\cdot)](\gamma\cdot)_{|\mathrm{cl}\Omega^*}$ is real analytic. Then Theorem 3.4 (iii) implies that the map from $]-\epsilon_{\tilde\Omega,r},\epsilon_{\tilde\Omega,r}[\times]-\delta',\delta'[\times\tilde\Gamma$ to $C^{m,\alpha}(\mathrm{cl}\Omega^*)$ which takes (ϵ,δ,γ) to $w^-[\partial\gamma\Omega,\Theta[\epsilon,\delta,\gamma](\gamma^{-1}\cdot)](\gamma\cdot)$ is real analytic.

Next we consider the integral operator in the right hand side of (35) and we proceed as in the proof of the analyticity of Λ in Theorem 3.4. Let Ω' be a bounded open subset of $\mathbb{R}^n$ of class C^1 such that $\mathrm{cl}\Omega'\subseteq\mathbb{R}^n\setminus q(\mathbb{Z}^n\setminus\{0\})$, $0\in\Omega'$. Clearly, $(\mathrm{cl}\Omega^*-\partial\Omega)$ is compact. Then possibly shrinking $\epsilon_{\tilde\Omega,r}$, we can assume that

$$\epsilon(t-s)\subseteq\Omega'\qquad\forall(\epsilon,\gamma,t,s)\in]-\epsilon_{\tilde\Omega,r},\epsilon_{\tilde\Omega,r}[\times\tilde\Gamma\times\mathrm{cl}\Omega^*\times\partial\Omega\,.$$

Then Theorem A.2 (i) of the Appendix implies that $DR_{\gamma q,n}(\epsilon\gamma(t-s))$ is analytic in the variable (ϵ,γ,t,s) in an open neighborhood of $]-\epsilon_{\tilde\Omega,r},\epsilon_{\tilde\Omega,r}[\times\tilde\Gamma\times\mathrm{cl}\Omega^*\times\partial\Omega$, and a result on integral operators with real analytic kernels and with no singularity (cf. [31, Prop. 4.1 (i)]) implies that the map from $]-\epsilon_{\tilde{\Omega},r},\epsilon_{\tilde{\Omega},r}[\times C^{m,\alpha}(\partial\Omega)$ to $C^{m,\alpha}(\mathrm{cl}\Omega^*)$ which takes (ϵ,γ,θ) to the function

$$\int_{\partial\Omega}(\gamma^{-1}\nu_\Omega(s))DR_{\gamma q,n}(\epsilon\gamma(t-s))\theta(s)\,d\sigma_s \qquad \forall t\in\mathrm{cl}\Omega^*\,,$$

is analytic. Then by the analyticity of $C[\cdot,\cdot,\cdot]$, we conclude that the map $V^{\sharp r}_{\Omega^*}$ is analytic. Then by Theorem 3.4 (iii) and by (25), we have

$$V^{\sharp r}_{\Omega^*}[0,0,\tilde{\gamma}](t)=w^-[\partial\tilde{\gamma}\Omega,\Theta[0,0,\tilde{\gamma}](\tilde{\gamma}^{-1}\cdot)](\tilde{\gamma}t)+C[0,0,\tilde{\gamma}]=\tilde{u}(t) \qquad \forall t\in\mathrm{cl}\Omega^*\,.$$

By Propositions A.4, A.5 and by Lemma A.7, the map from $]-\epsilon_{\tilde{\Omega},r},\epsilon_{\tilde{\Omega},r}[\times\mathbb{D}^+_n(\mathbb{R})$ to $C^{m,\alpha}(\mathrm{cl}\Omega^*)$ which takes (ϵ,γ) to $\mathcal{P}^r_{\Omega^*}[\epsilon,\gamma]$ is analytic.

Next we prove statement (iv). First we note that

$$\begin{aligned}
&\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}}u^\sharp(\epsilon,\delta,\gamma,x)\,dx\det\gamma\\
&\qquad=\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}}w[\partial\gamma\Omega_{p,\epsilon},S_{\gamma q,n},\Theta[\epsilon,\delta,\gamma](\gamma^{-1}\epsilon^{-1}(\cdot-\gamma p))](\gamma x)\det\gamma\,dx\\
&\qquad\quad+C[\epsilon,\delta,\gamma]m_n(Q\setminus\Omega_{p,\epsilon})\det\gamma+\delta^2\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}}P_{\gamma q,n}[\gamma Q,f(\gamma^{-1}\cdot)](\gamma x)\det\gamma\,dx
\end{aligned}$$

for all $(\epsilon,\delta,\gamma)\in]0,\epsilon'[\times]0,\delta'[\times\tilde{\Gamma}$. Next we note that

$$\begin{aligned}
&\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}}w[\partial\gamma\Omega_{p,\epsilon},S_{\gamma q,n},\Theta[\epsilon,\delta,\gamma](\gamma^{-1}\epsilon^{-1}(\cdot-\gamma p))](\gamma x)\det\gamma\,dx\\
&\quad=-\int_{\gamma Q\setminus\mathrm{cl}\gamma\Omega_{p,\epsilon}}\int_{\partial\gamma\Omega_{p,\epsilon}}\nu_{\gamma\Omega_{p,\epsilon}}(y)DS_{\gamma q,n}(x-y)\Theta[\epsilon,\delta,\gamma](\gamma^{-1}\epsilon^{-1}(y-\gamma p))\,d\sigma_y\,dx\\
&\quad=-\int_{\gamma Q\setminus\mathrm{cl}\gamma\Omega_{p,\epsilon}}\sum_{j=1}^n\frac{\partial}{\partial x_j}\int_{\partial\gamma\Omega_{p,\epsilon}}S_{\gamma q,n}(x-y)\Theta[\epsilon,\delta,\gamma](\gamma^{-1}\epsilon^{-1}(y-\gamma p))(\nu_{\gamma\Omega_{p,\epsilon}}(y))_j\,d\sigma_y\,dx\\
&\quad=\int_{\partial\gamma\Omega_{p,\epsilon}}\sum_{j=1}^n(\nu_{\gamma\Omega_{p,\epsilon}}(x))_j\int_{\partial\gamma\Omega_{p,\epsilon}}S_{\gamma q,n}(x-y)\Theta[\epsilon,\delta,\gamma](\gamma^{-1}\epsilon^{-1}(y-\gamma p))(\nu_{\gamma\Omega_{p,\epsilon}}(y))_j\,d\sigma_y\,d\sigma_x\\
&\quad=\sum_{j=1}^n\int_{\partial\Omega}(\gamma^{-1}\nu_\Omega(t))_j\int_{\partial\Omega}S_{\gamma q,n}(\epsilon\gamma(t-s))\Theta[\epsilon,\delta,\gamma](s)(\gamma^{-1}\nu_\Omega(s))_j\,d\sigma_s\,d\sigma_t\epsilon^{2n-2}(\det\gamma)^2\\
&\quad=\sum_{j=1}^n\int_{\partial\Omega}(\gamma^{-1}\nu_\Omega(t))_j\int_{\partial\Omega}S_n(\gamma(t-s))\Theta[\epsilon,\delta,\gamma](s)(\gamma^{-1}\nu_\Omega(s))_j\,d\sigma_s\,d\sigma_t\epsilon^n(\det\gamma)^2\\
&\qquad+\frac{\delta_{2,n}}{2\pi}\epsilon(\epsilon\log\epsilon)\sum_{j=1}^n\int_{\partial\Omega}(\gamma^{-1}\nu_\Omega(t))_j\,d\sigma_t\int_{\partial\Omega}\Theta[\epsilon,\delta,\gamma](s)(\gamma^{-1}\nu_\Omega(s))_j\,d\sigma_s(\det\gamma)^2\\
&\qquad+\sum_{j=1}^n\int_{\partial\Omega}(\gamma^{-1}\nu_\Omega(t))_j\int_{\partial\Omega}R_{\gamma q,n}(\epsilon\gamma(t-s))\Theta[\epsilon,\delta,\gamma](s)(\gamma^{-1}\nu_\Omega(s))_j\,d\sigma_s\,d\sigma_t\epsilon^{2n-2}(\det\gamma)^2\\
&\quad=\sum_{j=1}^n\int_{\partial\Omega}(\gamma^{-1}\nu_\Omega(t))_j\int_{\partial\Omega}S_n(\gamma(t-s))\Theta[\epsilon,\delta,\gamma](s)(\gamma^{-1}\nu_\Omega(s))_j\,d\sigma_s\,d\sigma_t\epsilon^n(\det\gamma)^2\\
&\qquad+\sum_{j=1}^n\int_{\partial\Omega}(\gamma^{-1}\nu_\Omega(t))_j\int_{\partial\Omega}R_{\gamma q,n}(\epsilon\gamma(t-s))\Theta[\epsilon,\delta,\gamma](s)(\gamma^{-1}\nu_\Omega(s))_j\,d\sigma_s\,d\sigma_t\epsilon^{2n-2}(\det\gamma)^2\,,
\end{aligned}$$

for all $(\epsilon,\gamma)\in]0,\epsilon'[\times\tilde{\Gamma}$. Thus it is natural to set

$$\begin{aligned}
J^\sharp_1[\epsilon,\gamma]\equiv&\sum_{j=1}^n\int_{\partial\Omega}(\gamma^{-1}\nu_\Omega(t))_j\int_{\partial\Omega}S_n(\gamma(t-s))\Theta[\epsilon,\delta,\gamma](s)(\gamma^{-1}\nu_\Omega(s))_j\,d\sigma_s\,d\sigma_t\epsilon^n(\det\gamma)\\
&+\sum_{j=1}^n\int_{\partial\Omega}(\gamma^{-1}\nu_\Omega(t))_j\int_{\partial\Omega}R_{\gamma q,n}(\epsilon\gamma(t-s))\Theta[\epsilon,\delta,\gamma](s)(\gamma^{-1}\nu_\Omega(s))_j\,d\sigma_s\,d\sigma_t\epsilon^{2n-2}(\det\gamma)\,,
\end{aligned}$$

for all $(\epsilon,\gamma)\in]-\epsilon',\epsilon'[\times\tilde{\Gamma}$. By the analyticity of $\mathcal{P}$ and by Proposition A.5 of the Appendix, the map $J_2^\sharp$ from $]-\epsilon',\epsilon'[\times\tilde{\Gamma}$ to $\mathbb{R}$ defined by

$$J_2^\sharp[\epsilon,\gamma]=-\epsilon^n\int_\Omega \mathcal{P}[\gamma](p+\epsilon s)\,ds \qquad \forall(\epsilon,\gamma)\in]-\epsilon',\epsilon'[\times\tilde{\Gamma}\,,$$

is real analytic and satisfies the equality

$$J_2^\sharp[\epsilon,\gamma]=-\int_{\Omega_{p,\epsilon}}\mathcal{P}[\gamma]\,dx \qquad \forall(\epsilon,\gamma)\in]0,\epsilon'[\times\tilde{\Gamma}\,.$$

Then we set

$$J^\sharp[\epsilon,\delta,\gamma]\equiv J_1^\sharp[\epsilon,\gamma]+C[\epsilon,\delta,\gamma](m_n(Q)-\epsilon^n m_n(\Omega))+\delta^2 J_2^\sharp[\epsilon,\gamma]$$

for all $(\epsilon,\delta,\gamma)\in]-\epsilon',\epsilon'[\times]-\delta',\delta'[\times\tilde{\Gamma}$. Clearly,

$$J^\sharp[0,0,\tilde{\gamma}]=C[0,0,\tilde{\gamma}]m_n(Q)=\tilde{c}m_n(Q)\,,$$

and thus statement (iv) holds true (see also Theorem 3.4 (iii).) □

Next we turn to consider the behavior of the energy integral of $u^\sharp(\epsilon,\delta,\gamma,\cdot)$, and we prove the following.

Proposition 4.2. *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $p\in Q$. Let Ω be as in (1). Let ϵ_0 be as in (2). Let $\tilde{\gamma}\in\mathbb{D}_n^+(\mathbb{R})$. Let f be as in (3). Let $g\in C^{m,\alpha}(\partial\Omega)$.*

Let $\tilde{u}$, $\tilde{c}$ be as in Theorem 3.3. Let ϵ', δ', $\tilde{\Gamma}$ be as in Theorem 3.4 (iii). Then there exists $\epsilon_e\in]0,\epsilon'[$ and a real analytic map $\mathcal{E}^\sharp$ from $]-\epsilon_e,\epsilon_e[\times]-\delta',\delta'[\times\tilde{\Gamma}$ to $\mathbb{R}$, and a real analytic map $\mathcal{P}_e^\sharp$ from $]-\epsilon_e,\epsilon_e[\times\tilde{\Gamma}$ to $\mathbb{R}$ such that

$$\int_{Q\setminus \mathrm{cl}\Omega_{p,\epsilon}}|D_x u^\sharp(\epsilon,\delta,\gamma,x)\gamma^{-1}|^2\,dx=\mathcal{E}^\sharp[\epsilon,\delta,\gamma]\epsilon^{n-2}+\delta^4\mathcal{P}_e^\sharp[\epsilon,\gamma] \tag{37}$$

for all $(\epsilon,\delta,\gamma)\in]0,\epsilon_e[\times]0,\delta[\times\tilde{\Gamma}$. Moreover

$$\begin{aligned}\mathcal{E}^\sharp[0,0,\tilde{\gamma}] &= \int_{\mathbb{R}^n\setminus\mathrm{cl}\Omega}|D_x\tilde{u}(x)\tilde{\gamma}^{-1}|^2\,dx\,, \\ \mathcal{P}_e^\sharp[0,\tilde{\gamma}] &= \int_Q\left|D_x\left(P_{\tilde{\gamma}q,n}[\tilde{\gamma}Q,f(\tilde{\gamma}^{-1}\cdot)](\tilde{\gamma}x)\right)\tilde{\gamma}^{-1}\right|^2\,dx\,.\end{aligned} \tag{38}$$

Proof. Clearly, we have

$$\begin{aligned}&\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}}|D_x u^\sharp(\epsilon,\delta,\gamma,x)\gamma^{-1}|^2\,dx \\ &\quad=\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}}\left|D_x\omega^\sharp(\epsilon,\delta,\gamma,x)\gamma^{-1}+\delta^2 D_x\left(P_{\gamma q,n}[\gamma Q,f(\gamma^{-1}\cdot)](\gamma x)\right)\gamma^{-1}\right|^2 dx \\ &\quad=\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}}|D_x\omega^\sharp(\epsilon,\delta,\gamma,x)\gamma^{-1}|^2\,dx \\ &\qquad+2\delta^2\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}}D_x\omega^\sharp(\epsilon,\delta,\gamma,x)\gamma^{-1}\left[D_x\left(P_{\gamma q,n}[\gamma Q,f(\gamma^{-1}\cdot)](\gamma x)\right)\gamma^{-1}\right]^t\,dx \\ &\qquad+\delta^4\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}}\left|D_x\left(P_{\gamma q,n}[\gamma Q,f(\gamma^{-1}\cdot)](\gamma x)\right)\gamma^{-1}\right|^2 dx \\ &\quad=\det\gamma^{-1}\int_{\gamma Q\setminus\gamma\mathrm{cl}\Omega_{p,\epsilon}}\mathrm{div}\left[\omega^\sharp(\epsilon,\delta,\gamma,\gamma^{-1}y)D_y\left(\omega^\sharp(\epsilon,\delta,\gamma,\gamma^{-1}y)\right)\right]\,dy \\ &\qquad+2\delta^2\det\gamma^{-1}\int_{\gamma Q\setminus\gamma\mathrm{cl}\Omega_{p,\epsilon}}\mathrm{div}\left[P_{\gamma q,n}[\gamma Q,f(\gamma^{-1}\cdot)](y)D(\omega^\sharp(\epsilon,\delta,\gamma,\gamma^{-1}y))\right]\,dy \\ &\qquad+\delta^4\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}}\left|D_x\left(P_{\gamma q,n}[\gamma Q,f(\gamma^{-1}\cdot)](\gamma x)\right)\gamma^{-1}\right|^2 dx\,,\end{aligned} \tag{39}$$

for all $(\epsilon,\delta,\gamma)\in]0,\epsilon_0[\times]0,+\infty[\times\tilde{\Gamma}$ (cf. (14).) We now consider separately each term in the right hand side of (39). By the Divergence Theorem and by the q-periodicity of $\omega^\sharp(\epsilon,\delta,\gamma,\cdot)$, we have

$$\det\gamma^{-1}\int_{\gamma Q\setminus\gamma\mathrm{cl}\Omega_{p,\epsilon}}\mathrm{div}\left[\omega^\sharp(\epsilon,\delta,\gamma,\gamma^{-1}y)D_y\left(\omega^\sharp(\epsilon,\delta,\gamma,\gamma^{-1}y)\right)\right]\,dy$$

$$
\begin{aligned}
&= -\det\gamma^{-1}\int_{\partial\gamma\Omega_{p,\epsilon}} \omega^\sharp(\epsilon,\delta,\gamma,\gamma^{-1}y)\frac{\partial}{\partial\nu_{\gamma\Omega_{p,\epsilon}}(y)}\left(\omega^\sharp(\epsilon,\delta,\gamma,\gamma^{-1}y)\right)d\sigma_y\\
&= -\det\gamma^{-1}\int_{\partial\gamma\Omega_{p,\epsilon}} \omega^\sharp(\epsilon,\delta,\gamma,\gamma^{-1}y)\nu_{\gamma\Omega_{p,\epsilon}}(y)D_y\left(\omega^\sharp(\epsilon,\delta,\gamma,\gamma^{-1}y)\right)d\sigma_y\\
&= -\int_{\partial\Omega_{p,\epsilon}} \omega^\sharp(\epsilon,\delta,\gamma,x)(\gamma^{-1}\nu_{\Omega_{p,\epsilon}}(x))(D_x\omega^\sharp(\epsilon,\delta,\gamma,x)\gamma^{-1})\,d\sigma_x\\
&= -\int_{\partial\Omega_{p,\epsilon}} g(\epsilon^{-1}(x-p))(\gamma^{-1}\nu_{\Omega_{p,\epsilon}}(x))(D_x\omega^\sharp(\epsilon,\delta,\gamma,x)\gamma^{-1})\,d\sigma_x\\
&+\delta^2\int_{\partial\Omega_{p,\epsilon}} P_{\gamma q,n}[\gamma Q,f(\gamma^{-1}\cdot)](\gamma x)(\gamma^{-1}\nu_{\Omega_{p,\epsilon}}(x))(D_x\omega^\sharp(\epsilon,\delta,\gamma,x)\gamma^{-1})\,d\sigma_x\\
&= -\int_{\partial\Omega} g(t)(\gamma^{-1}\nu_\Omega(t))(D_t(\omega^\sharp(\epsilon,\delta,\gamma,p+\epsilon t)\gamma^{-1}))\,d\sigma_t\epsilon^{n-2}\\
&+\delta^2\int_{\partial\Omega} P_{\gamma q,n}[\gamma Q,f(\gamma^{-1}\cdot)](\gamma(p+\epsilon t))(\gamma^{-1}\nu_\Omega(t))(D_t(\omega^\sharp(\epsilon,\delta,\gamma,p+\epsilon t)\gamma^{-1}))\,d\sigma_t\epsilon^{n-2},
\end{aligned}
$$

for all $(\epsilon,\delta,\gamma)\in]0,\epsilon'[\times]0,\delta'[\times\tilde\Gamma$. Now let $R\in]0,+\infty[$ be such that $\mathrm{cl}\Omega\subseteq\mathbb{B}_n(0,R)$, $\tilde\Omega\equiv\mathbb{B}_n(0,R)\setminus\mathrm{cl}\Omega$. Then we take $\epsilon_{\tilde\Omega,r}$, $V^{\sharp r}_{\tilde\Omega}$, $\mathcal{P}^r_{\tilde\Omega}$ as in Proposition 4.1 (iii), and we set

$$
\begin{aligned}
\mathcal{E}^\sharp_1[\epsilon,\delta,\gamma]\equiv&-\int_{\partial\Omega} g(t)(\gamma^{-1}\nu_\Omega(t))(D_t(V^{\sharp r}_{\tilde\Omega}[\epsilon,\delta,\gamma](t))\gamma^{-1})d\sigma_t\\
&+\delta^2\int_{\partial\Omega}\mathcal{P}^r_{\tilde\Omega}[\epsilon,\gamma](t)(\gamma^{-1}\nu_\Omega(t))(D_t(V^{\sharp r}_{\tilde\Omega}[\epsilon,\delta,\gamma](t))\gamma^{-1})d\sigma_t
\end{aligned}
$$

for all $(\epsilon,\delta,\gamma)\in]-\epsilon_{\tilde\Omega,r},\epsilon_{\tilde\Omega,r}[\times]-\delta',\delta'[\times\tilde\Gamma$. Then we have

$$
\det\gamma^{-1}\int_{\gamma Q\setminus\gamma\mathrm{cl}\Omega_{p,\epsilon}}\mathrm{div}\left[\omega^\sharp(\epsilon,\delta,\gamma,\gamma^{-1}y)D_y\left(\omega^\sharp(\epsilon,\delta,\gamma,\gamma^{-1}y)\right)\right]\,dy=\mathcal{E}^\sharp_1[\epsilon,\delta,\gamma]\epsilon^{n-2}
$$

for all $(\epsilon,\delta,\gamma)\in]0,\epsilon_{\tilde\Omega,r}[\times]0,\delta'[\times\tilde\Gamma$. By Proposition 4.1 (iii), $\mathcal{E}_1$ is analytic. Moreover,

$$
\begin{aligned}
\mathcal{E}^\sharp_1[0,0,\tilde\gamma]=&-\int_{\partial\Omega} g(t)(\tilde\gamma^{-1}\nu_\Omega(t))(D_t(V^{\sharp r}_{\tilde\Omega}[0,0,\tilde\gamma](t))\tilde\gamma^{-1})d\sigma_t\\
&=-\int_{\partial\Omega} g(t)(\tilde\gamma^{-1}\nu_\Omega(t))(D\tilde u(t)\tilde\gamma^{-1})d\sigma_t\\
&=-\int_{\partial\tilde\gamma\Omega} g(\tilde\gamma^{-1}x)\nu_{\tilde\gamma\Omega}(x)D_x(\tilde u(\tilde\gamma^{-1}x))\,d\sigma_x\det\tilde\gamma^{-1}\\
&=-\int_{\partial\tilde\gamma\Omega}(g(\tilde\gamma^{-1}x)-\tilde c)\nu_{\tilde\gamma\Omega}(x)D_x(\tilde u(\tilde\gamma^{-1}x)-\tilde c)\,d\sigma_x\det\tilde\gamma^{-1}\\
&\quad-\int_{\partial\tilde\gamma\Omega}\tilde c\nu_{\tilde\gamma\Omega}(x)D_x(\tilde u(\tilde\gamma^{-1}x)-\tilde c)\,d\sigma_x\det\tilde\gamma^{-1}\\
&=\int_{\mathbb{R}^n\setminus\mathrm{cl}\tilde\gamma\Omega}|D_x(\tilde u(\tilde\gamma^{-1}x)-\tilde c)|^2\,dx\det\tilde\gamma^{-1}\\
&=\int_{\mathbb{R}^n\setminus\mathrm{cl}\Omega}|D_x\tilde u(x)\tilde\gamma^{-1}|^2\,dx\,.
\end{aligned}
$$

Indeed, $\tilde u(\tilde\gamma^{-1}x)-\tilde c$ is harmonic at infinity and equals $g(\tilde\gamma^{-1}x)-\tilde c$ on $\partial\Omega$ (cf. Folland [21, Props. 2.74, 2.75, proof of Prop. 3.4].) Next we note that

$$
\begin{aligned}
&2\delta^2\det\gamma^{-1}\int_{\gamma Q\setminus\gamma\mathrm{cl}\Omega_{p,\epsilon}}\mathrm{div}\left\{P_{\gamma q,n}[\gamma Q,f(\gamma^{-1}\cdot)](y)D_y\left[\omega^\sharp(\epsilon,\delta,\gamma,\gamma^{-1}y)\right]\right\}\,dy\\
&\quad=-2\delta^2\det\gamma^{-1}\int_{\partial\gamma\Omega_{p,\epsilon}}P_{\gamma q,n}[\gamma Q,f(\gamma^{-1}\cdot)](y)\frac{\partial}{\partial\nu_{\gamma\Omega_{p,\epsilon}}(y)}\left(\omega^\sharp(\epsilon,\delta,\gamma,\gamma^{-1}y)\right)d\sigma_y\\
&\quad=-2\delta^2\det\gamma^{-1}\int_{\partial\gamma\Omega_{p,\epsilon}}P_{\gamma q,n}[\gamma Q,f(\gamma^{-1}\cdot)](y)\nu_{\gamma\Omega_{p,\epsilon}}(y)D_y\left[\omega^\sharp(\epsilon,\delta,\gamma,\gamma^{-1}y)\right]\,d\sigma_y\\
&\quad=-2\delta^2\int_{\partial\Omega_{p,\epsilon}}P_{\gamma q,n}[\gamma Q,f(\gamma^{-1}\cdot)](\gamma x)(\gamma^{-1}\nu_{\Omega_{p,\epsilon}}(x))(D_x\omega^\sharp(\epsilon,\delta,\gamma,x)\gamma^{-1})\,d\sigma_x
\end{aligned}
$$

$$= -2\delta^2 \int_{\partial\Omega} P_{\gamma q,n}[\gamma Q, f(\gamma^{-1}\cdot)](\gamma(p+\epsilon t))$$
$$\times(\gamma^{-1}\nu_\Omega(t)) \left(D_t \left[\omega^\sharp(\epsilon,\delta,\gamma,p+\epsilon t)\right]\gamma^{-1}\right) d\sigma_t \epsilon^{n-2}\,,$$

for all $(\epsilon,\delta,\gamma) \in]0,\epsilon'[\times]0,\delta'[\times\tilde{\Gamma}$. Then we set

$$\mathcal{E}_2^\sharp[\epsilon,\delta,\gamma] \equiv -2\delta^2 \int_{\partial\Omega} \mathcal{P}^r_{\tilde{\Omega}}[\epsilon,\gamma](t)(\gamma^{-1}\nu_\Omega(t)) \left(D_t \left[V^{\sharp r}_{\tilde{\Omega}}[\epsilon,\delta,\gamma](t)\right]\gamma^{-1}\right) d\sigma_t\,,$$

for all $(\epsilon,\delta,\gamma) \in]-\epsilon_{\tilde{\Omega},r},\epsilon_{\tilde{\Omega},r}[\times]-\delta',\delta'[\times\tilde{\Gamma}$. Then we have

$$2\delta^2 \det\gamma^{-1} \int_{\gamma Q\setminus\gamma\mathrm{cl}\Omega_{p,\epsilon}} \mathrm{div}\left\{P_{\gamma q,n}[\gamma Q, f(\gamma^{-1}\cdot)](y) D\left[\omega^\sharp(\epsilon,\delta,\gamma,\gamma^{-1}y)\right]\right\} dy$$
$$= \mathcal{E}_2^\sharp[\epsilon,\delta,\gamma]\epsilon^{n-2}\,,$$

for all $(\epsilon,\delta,\gamma) \in]0,\epsilon_{\tilde{\Omega},r}[\times]0,\delta'[\times\tilde{\Gamma}$. By Proposition 4.1 (iii), $\mathcal{E}_2$ is analytic. Moreover,

$$\mathcal{E}_2^\sharp[0,0,\tilde{\gamma}] = 0\,.$$

By Proposition 4.1 (i) and by Lemma A.8, there exists a real analytic map $\mathcal{P}_e^\sharp$ from $]-\epsilon_0,\epsilon_0[\times\tilde{\Gamma}$ to $\mathbb{R}$ such that

$$\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}} |D_x\left(P_{\gamma q,n}[\gamma Q, f(\gamma^{-1}\cdot)](\gamma x)\right)\gamma^{-1}|^2\,dx = \mathcal{P}_e^\sharp[\epsilon,\gamma] \qquad \forall(\epsilon,\gamma)\in]0,\epsilon_0[\times\tilde{\Gamma}\,.$$

Moreover, the second equality in (38) holds true. Then by setting $\epsilon_e \equiv \min\{\epsilon_{\tilde{\Omega},r},\epsilon_0\}$ and

$$\mathcal{E}^\sharp[\epsilon,\delta,\gamma] \equiv \mathcal{E}_1^\sharp[\epsilon,\delta,\gamma] + \mathcal{E}_2^\sharp[\epsilon,\delta,\gamma]\,,$$

for all $(\epsilon,\delta,\gamma) \in]-\epsilon_e,\epsilon_e[\times]-\delta',\delta'[\times\tilde{\Gamma}$, equalities (37) and (38) hold true. □

The function $u(\epsilon,\delta,\gamma,\cdot)$ can be extended to the whole of $\mathbb{R}^n$ by setting it equal to zero outside of its domain $\mathrm{cl}\mathbb{S}(\epsilon,\delta,\gamma)^-$. Then one can ask whether the extension of $u(\epsilon,\delta,\gamma,\cdot)$ has a limit as (ϵ,δ,γ) converges to $(0,0,\tilde{\gamma})$, a question which we answer below. To do so, we introduce a notation for the extension of $u(\epsilon,\delta,\gamma,\cdot)$.

Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $p \in Q$. Let Ω be as in (1). Let ϵ_0 be as in (2). Let $(\epsilon,\delta,\gamma) \in]0,\epsilon_0[\times]0,+\infty[\times\mathbb{D}_n^+(\mathbb{R})$. If v is a function from $\mathrm{cl}\mathbb{S}(\epsilon,\delta,\gamma)^-$ to $\mathbb{R}$, then we denote by $\mathbf{E}_{(\epsilon,\delta,\gamma)}[v]$ the function from $\mathbb{R}^n$ to $\mathbb{R}$ defined by

$$\mathbf{E}_{(\epsilon,\delta,\gamma)}[v](x) \equiv \begin{cases} v(x) & \forall x\in\mathrm{cl}\mathbb{S}(\epsilon,\delta,\gamma)^-\,, \\ 0 & \forall x\in\mathbb{R}^n\setminus\mathrm{cl}\mathbb{S}(\epsilon,\delta,\gamma)^-\,. \end{cases}$$

Then we have the following.

Proposition 4.3. *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $p\in Q$. Let Ω be as in (1). Let ϵ_0 be as in (2). Let $\tilde{\gamma}\in\mathbb{D}_n^+(\mathbb{R})$. Let f be as in (3). Let $g\in C^{m,\alpha}(\partial\Omega)$.*

Let $\{(\varepsilon_j,\delta_j,\gamma_j)\}_{j\in\mathbb{N}}$ be a sequence in $]0,\epsilon_0[\times]0,+\infty[\times\mathbb{D}_n^+(\mathbb{R})$ which converges to $(0,0,\tilde{\gamma})$. Let $r\in[1,+\infty[$. Then

$$\lim_{j\to\infty} \mathbf{E}_{(\varepsilon_j,1,I)}[u^\sharp(\varepsilon_j,\delta_j,\gamma_j,\cdot)] = \tilde{c} \qquad \text{in } L^r(Q)\,,$$

where $\tilde{c}$ is as in Theorem 3.3.

Proof. Let ϵ', δ' be as in Theorem 3.4 (iii). Possibly neglecting a finite number of indexes j, we can assume that $\varepsilon_j<\epsilon'$, $\delta_j<\delta'$ for all $j\in\mathbb{N}$. By formula (9), we have

$$\max_{x\in\mathrm{cl}\mathbb{S}[\Omega_{p,\varepsilon_j}]^-} |u^\sharp(\varepsilon_j,\delta_j,\gamma_j,x)|$$
$$\leq \max_{x\in\mathrm{cl}\mathbb{S}[\Omega_{p,\varepsilon_j}]^-} |\omega^\sharp(\varepsilon_j,\delta_j,\gamma_j,x)| + \delta_j^2 \max_{x\in\mathbb{R}^n} |P_{\gamma_j q,n}[\gamma_j Q, f(\gamma_j^{-1}\cdot)](\gamma_j x)|\,.$$

Let $\tilde{Q}\equiv\Pi_{l=1}^n]-q_{ll}/2,q_{ll}/2[$. Since $\left(\gamma_j Q\setminus\bigcup_{z\in\{0,1\}^n}(\gamma_j q z+\gamma_j\tilde{Q}\right)$ has measure zero and $S_{\gamma_j q,n}(\cdot)$ is $\gamma_j q$-periodic, we have

$$\int_{\gamma_j Q} |S_{\gamma_j q,n}(x-y)|\,dy \leq \sum_{z\in\{0,1\}^n} \int_{\gamma_j q z+\gamma_j\tilde{Q}} |S_{\gamma_j q,n}(x-y)|\,dy$$

$$\leq 2^n \int_{\gamma_j \tilde{Q}} |S_{\gamma_j q,n}(-y)|\, dy = 2^n \int_{\tilde{Q}} |S_{\gamma_j q,n}(\gamma_j \xi)|\, d\xi \det \gamma_j \,.$$

Hence, the Maximum Principle implies that

$$\max_{x\in \mathrm{cl}\mathbb{S}[\Omega_{p,\varepsilon_j}]^-} |u^\sharp(\varepsilon_j,\delta_j,\gamma_j,x)| \tag{40}$$
$$\leq \|g_j\|_{C^0(\partial\Omega)} + 2\delta_j^2 \max_{x\in \mathrm{cl}\gamma_j Q} |f(\gamma_j^{-1}x)| 2^n \int_{\tilde{Q}} |y|^{-\lambda}\, dy \sup_{x\in \mathrm{cl}\tilde{Q}\setminus\{0\}} |S_{\gamma_j q,n}(\gamma_j x)|\, |x|^\lambda \,.$$

Moreover,

$$\sup_{x\in \mathrm{cl}\tilde{Q}\setminus\{0\}} |S_{\gamma_j q,n}(\gamma_j x)|\, |x|^\lambda \tag{41}$$
$$\leq \sup_{x\in \mathrm{cl}\tilde{Q}\setminus\{0\}} |S_n(\gamma_j x)|\, |x|^\lambda + \sup_{x\in \mathrm{cl}\tilde{Q}\setminus\{0\}} |R_{\gamma_j q,n}(\gamma_j x)|\, |x|^\lambda$$
$$\leq \left(\sup_{\xi\in \gamma_j \mathrm{cl}\tilde{Q}\setminus\{0\}} |S_n(\xi)|\, |\xi|^\lambda \right) \sup_{x\in \mathrm{cl}\tilde{Q}\setminus\{0\}} \frac{|x|^\lambda}{|\gamma_j x|^\lambda} + \sup_{x\in \mathrm{cl}\tilde{Q}\setminus\{0\}} |R_{\gamma_j q,n}(\gamma_j x)|\, |x|^\lambda \,,$$

(see also [18].) Now let $\tilde{\Gamma}$ be a ball in $\mathbb{D}_n^+(\mathbb{R})$ with center $\tilde{\gamma}$ and closure contained in $\mathbb{D}_n^+(\mathbb{R})$. Then there exists $a \in]1,+\infty[$ such that

$$\frac{1}{a}|\gamma\xi| \leq |\xi| \leq a|\gamma\xi| \qquad \forall \gamma \in \tilde{\Gamma}\,,\ \forall \xi \in \mathbb{R}^n \,. \tag{42}$$

Then by inequalities (40)–(42), and by Theorem A.2 (i), there exists $M \in]0,+\infty[$ such that

$$\max_{x\in \mathrm{cl}\mathbb{S}[\Omega_{p,\varepsilon_j}]^-} |u^\sharp(\varepsilon_j,\delta_j,\gamma_j,x)| \leq M \,,$$

for all $j \in \mathbb{N}$ such that $\gamma_j \in \tilde{\Gamma}$. As a consequence,

$$|\mathbf{E}_{(\varepsilon_j,1,I)}[u^\sharp(\varepsilon_j,\delta_j,\gamma_j,\cdot)](x)| \leq M \qquad \forall x \in \mathbb{R}^n \,,$$

for all $j \in \mathbb{N}$ such that $\gamma_j \in \tilde{\Gamma}$. By Proposition 4.1 (ii),

$$\lim_{j\to\infty} \mathbf{E}_{(\varepsilon_j,1,I)}[u^\sharp(\varepsilon_j,\delta_j,\gamma_j,\cdot)](x) = \tilde{c} \qquad \text{for almost all } x \in \mathbb{R}^n \,.$$

Hence, the Dominated Convergence Theorem in the set Q implies the validity of the statement. □

5 A functional analytic representation theorem for the solution of problem (4)

We now exploit the results on the solution of the auxiliary problem (7) in order to analyze the behavior of the solution of problem (4). We first show the validity of the following convergence result in Lebesgue spaces.

Theorem 5.1. *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$. Let $p \in Q$. Let Ω be as in (1). Let ϵ_0 be as in (2). Let $\tilde{\gamma} \in \mathbb{D}_n^+(\mathbb{R})$. Let f be as in (3). Let $g \in C^{m,\alpha}(\partial\Omega)$. Let $\tilde{u}$, $\tilde{c}$ be as in Theorem 3.3. Let $\{(\varepsilon_j,\delta_j,\gamma_j)\}_{j\in\mathbb{N}}$ be a sequence in $]0,\epsilon_0[\times]0,+\infty[\times\mathbb{D}_n^+(\mathbb{R})$ which converges to $(0,0,\tilde{\gamma})$. Let $r \in [1,+\infty[$. Let V be a bounded open subset of $\mathbb{R}^n$. Then*

$$\lim_{j\to\infty} \mathbf{E}_{(\varepsilon_j,\delta_j,\gamma_j)}[u(\varepsilon_j,\delta_j,\gamma_j,\cdot)] = \tilde{c} \qquad \text{in } L^r(V)\,. \tag{43}$$

Proof. We first show that there exists a constant $C \in]0,+\infty[$ such that

$$\|\mathbf{E}_{(\varepsilon_j,\delta_j,\gamma_j)}[u(\varepsilon_j,\delta_j,\gamma_j,\cdot)] - \tilde{c}\|_{L^r(V)} \leq C \|\mathbf{E}_{(\varepsilon_j,1,I)}[u^\sharp(\varepsilon_j,\delta_j,\gamma_j,\cdot)] - \tilde{c}\|_{L^r(Q)} \qquad \forall j \in \mathbb{N}\,. \tag{44}$$

We do so by exploiting a slight variant of a known dilation argument for periodic functions (see Braides and De Franceschi [6, ex. 27, p. 20].) Let γ_{j*} and γ_j^* denote the minimum and the maximum eigenvalue of the matrix γ_j, respectively. Since the sequence $\{\gamma_j\}_{j\in\mathbb{N}}$ is convergent in $\mathbb{D}_n^+(\mathbb{R})$, we have

$$0 < \mu_* \equiv \inf_{j\in\mathbb{N}} \gamma_{j*}\,, \qquad \mu^* \equiv \sup_{j\in\mathbb{N}} \gamma_j^* < \infty\,.$$

Since V is bounded, there exists $s \in \mathbb{N}$ such that

$$V \subseteq s\tilde{Q} \qquad \text{where } \tilde{Q} \equiv \Pi_{l=1}^{n}]-q_{ll}/2, q_{ll}/2[\,.$$

Then the q-periodicity of $\mathbf{E}_{(\varepsilon_j,1,I)}[u^\sharp(\varepsilon_j,\delta_j,\gamma_j,\cdot)]$ and equality (8) imply that

$$\begin{aligned}
\int_V |\mathbf{E}_{(\varepsilon_j,\delta_j,\gamma_j)}[u(\varepsilon_j,\delta_j,\gamma_j,\cdot)](x) - \tilde{c}|^r\,dx &\le \int_{s\tilde{Q}} |\mathbf{E}_{(\varepsilon_j,\delta_j,\gamma_j)}[u(\varepsilon_j,\delta_j,\gamma_j,\cdot)](x) - \tilde{c}|^r\,dx \\
&\le \int_{s\delta_j^{-1}\gamma_j^{-1}\tilde{Q}} |\mathbf{E}_{(\varepsilon_j,1,I)}[u^\sharp(\varepsilon_j,\delta_j,\gamma_j,\cdot)](y) - \tilde{c}|^r\,dy\delta_j^n \det\gamma_j \\
&\le \int_{s([\delta_j^{-1}]+1)([\mu_*^{-1}]+1)\tilde{Q}} |\mathbf{E}_{(\varepsilon_j,1,I)}[u^\sharp(\varepsilon_j,\delta_j,\gamma_j,\cdot)](y) - \tilde{c}|^r\,dy\delta_j^n(\mu^*)^n \\
&= s^n([\delta_j^{-1}]+1)^n([\mu_*^{-1}]+1)^n\delta_j^n(\mu^*)^n \int_{\tilde{Q}} |\mathbf{E}_{(\varepsilon_j,1,I)}[u^\sharp(\varepsilon_j,\delta_j,\gamma_j,\cdot)](y) - \tilde{c}|^r\,dy \\
&\le C^r \int_Q |\mathbf{E}_{(\varepsilon_j,1,I)}[u^\sharp(\varepsilon_j,\delta_j,\gamma_j,\cdot)](y) - \tilde{c}|^r\,dy
\end{aligned}$$

where

$$C \equiv \left\{ s^n([\mu_*^{-1}]+1)^n(\mu^*)^n \sup_{j\in\mathbb{N}} \left[([\delta_j^{-1}]+1)^n\delta_j^n\right] \right\} < \infty\,,$$

and where the brackets denote the integer part. Hence, inequality (44) follows. Then inequality (44) and Proposition 4.3 imply the validity of the limiting relation (43). □

The result above is akin to those obtained by variational methods, although here the methods are completely different. We now exploit our methods to describe the convergence of $u(\epsilon,\delta,\gamma,\cdot)$ as (ϵ,δ,γ) tends to $(0,0,\tilde{\gamma})$ in terms of analytic functions evaluated at specific values of (ϵ,δ,γ) in the spirit of the point of view of the present paper.

We first note that if $\tilde{\Omega}$ is a nonempty open subset of $\mathbb{R}^n$, then

$$\tilde{\Omega} \cap \left(\mathbb{R}^n \setminus \mathrm{cl}\mathbb{S}(\epsilon,\delta,\gamma)^-\right) \neq \emptyset\,,$$

whenever (ϵ,δ,γ) is sufficiently close to $(0,0,\tilde{\gamma})$. Hence, $u(\epsilon,\delta,\gamma,\cdot)$ is not defined in the whole of $\tilde{\Omega}$ for (ϵ,δ,γ) is sufficiently close to $(0,0,\tilde{\gamma})$, and we cannot hope to describe the behavior of $u(\epsilon,\delta,\gamma,\cdot)$ as we did for $u^\sharp(\epsilon,\delta,\gamma,\cdot)$ in Proposition 4.1. Hence, we must resort to different avenues.

We first fix $r \in [1,+\infty[$ and we identify $\mathbf{E}_{(\epsilon,\delta,\gamma)}[u(\epsilon,\delta,\gamma,\cdot)]$ with the corresponding functional in the dual of the space of functions of $L^{r'}(\mathbb{R}^n)$ with compact support, where r' is the conjugate exponent to r, and we would like to describe the 'weak' behavior of $\mathbf{E}_{(\epsilon,\delta,\gamma)}[u(\epsilon,\delta,\gamma,\cdot)]$ as (ϵ,δ,γ) tends to $(0,0,\tilde{\gamma})$ in terms of analytic maps. More precisely, we would like to describe the behavior of the integral

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,\delta,\gamma)}[u(\epsilon,\delta,\gamma,\cdot)]\phi\,dx \tag{45}$$

as (ϵ,δ,γ) tends to $(0,0,\tilde{\gamma})$ in terms of analytic maps, for all elements ϕ with compact support of $L^{r'}(\mathbb{R}^n)$. At the moment however, we cannot do so for all elements ϕ with compact support of $L^{r'}(\mathbb{R}^n)$, but only for all the elements ϕ which belong to a certain dense subspace $\mathcal{T}_q$ of $L^{r'}(\mathbb{R}^n)$ of functions with compact support, which we now turn to introduce by means of the following.

Proposition 5.2. *Let $\mathcal{T}_q$ be the vector subspace of $L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ generated by the set of functions*

$$\{\chi_{sQ+y} : (s,y) \in (\mathbb{Q}\cap]0,+\infty[) \times \mathbb{R}^n\}\,.$$

(i) If $r \in [1,+\infty[$, then the space $\mathcal{T}_q$ is dense in $L^r(\mathbb{R}^n)$.

(ii) If $\phi \in \mathcal{T}_q$, then there exist $y_1,\ldots,y_r \in \mathbb{R}^n$, and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$, and $s \in \mathbb{Q}\cap]0,+\infty[$ such that

$$\phi(x) = \sum_{l=1}^{r} \lambda_l \chi_{y_l+sQ}(x) \qquad \text{for a.a. } x \in \mathbb{R}^n\,.$$

Proof. We first prove statement (i). By the density of $C_c^\infty(\mathbb{R}^n)$ in $L^r(\mathbb{R}^n)$, it suffices to show that if $\phi \in C_c^\infty(\mathbb{R}^n)$ in $L^r(\mathbb{R}^n)$, then there exists a sequence $\{\phi_l\}_{l\in\mathbb{N}}$ in $\mathcal{T}_q$, which converges to ϕ in $L^r(\mathbb{R}^n)$. Let $\phi \in C_c^\infty(\mathbb{R}^n)$. We first assume that $\operatorname{supp}\phi \subseteq]0,+\infty[^n$. Let $\tilde{s} \in \mathbb{Q}\cap]0,+\infty[$ be such that $\operatorname{supp}\phi \subseteq \tilde{s}Q$. Then we define the sequence $\{\phi_l\}_{l\in\mathbb{N}}$ by setting

$$\phi_0(x) \equiv 0\,, \qquad \phi_l(x) \equiv \sum_{z\in\{0,\dots,l-1\}^n} \phi(\tilde{s}l^{-1}qz)\chi_{\tilde{s}l^{-1}Q+\tilde{s}l^{-1}qz}(x) \qquad \forall x \in \mathbb{R}^n\,,$$

for all $l \in \mathbb{N}\setminus\{0\}$. Clearly, $\phi_l \in \mathcal{T}_q$ for all $l \in \mathbb{N}$. Moreover, the continuity of ϕ implies that $\lim_{l\to\infty}\phi_l(x) = \phi(x)$ for almost all $x \in \mathbb{R}^n$. Since

$$|\phi_l(x)| \le \chi_{\tilde{s}Q}(x)\sup_{\mathbb{R}^n}|\phi| \qquad \forall x \in \mathbb{R}^n\,,$$

for all $l \in \mathbb{N}$, the Dominated Convergence Theorem implies that

$$\lim_{l\to\infty}\phi_l = \phi \qquad \text{in } L^r(\mathbb{R}^n)\,.$$

Next we release the assumption that the support of ϕ is contained in $]0,+\infty[^n$. Clearly, there exists $\tilde{x} \in \mathbb{R}^n$ depending on ϕ such that the translate function $\tilde{\phi}(\cdot) \equiv \phi(\cdot - \tilde{x})$ has support in $]0,+\infty[^n$. Then the case above implies the existence of a sequence $\{\tilde{\phi}_l\}_{l\in\mathbb{N}}$ in $\mathcal{T}_q$ such that $\lim_{l\to\infty}\tilde{\phi}_l = \tilde{\phi}$ in $L^r(\mathbb{R}^n)$. Hence, $\lim_{l\to\infty}\tilde{\phi}_l(\cdot+\tilde{x}) = \phi(\cdot)$ in $L^r(\mathbb{R}^n)$. Since $\tilde{\phi}_l(\cdot+\tilde{x}) \in \mathcal{T}_q$ for all $l \in \mathbb{N}$, the proof of statement (i) is complete.

Next we turn to statement (ii). By assumption, there exist $\xi_1, \dots, \xi_m \in \mathbb{R}^n$, and $\mu_1,\dots, \mu_m \in \mathbb{R}$, and $s_1, \dots, s_m \in \mathbb{Q}\cap]0,+\infty[$ such that

$$\phi(x) = \sum_{l=1}^m \mu_l \chi_{\xi_l+s_lQ}(x) \qquad \forall x \in \mathbb{R}^n\,.$$

Since s_l is rational, then there exist $u_l \in \mathbb{N}$ and $v_l \in \mathbb{N}\setminus\{0\}$ such that $s_l = \frac{u_l}{v_l}$, for each $l \in \{1,\dots,m\}$. Now let $b \equiv \Pi_{l=1}^m v_l$. Each n-dimensional interval $\xi_l + s_lQ$ can be written as a disjoint union of a set of measure zero and of $\left(u_l\frac{b}{v_l}\right)^n$ translations of the n-dimensional interval $b^{-1}Q$. Then statement (ii) holds with $s \equiv b^{-1}$. □

Next we turn to analyze the behavior of the integral in (45) with $\phi \in \mathcal{T}_q$ as (ϵ,δ,γ) tends to $(0,0,\tilde{\gamma})$ for some $\tilde{\gamma} \in \mathbb{D}_n^+(\mathbb{R})$. In the spirit of this paper, we represent the integral in (45) in terms of analytic maps evaluated at specific values of (ϵ,δ,γ). Namely, at values of (ϵ,δ,γ) such that the periodic cell $\delta\gamma Q$ is a certain integer fraction of the periodicity cell Q. More precisely, we require that δ equals the reciprocal of some integer $l \in \mathbb{N}\setminus\{0\}$ and that the entry γ_{jj} of γ equals the reciprocal of some integer number $a_{jj} \in \mathbb{N}\setminus\{0\}$ for all $j \in \{1,\dots,n\}$ and we prove the following.

Theorem 5.3. *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$. Let $p \in Q$. Let Ω be as in (1). Let ϵ_0 be as in (2). Let $a_{jj} \in \mathbb{N}\setminus\{0\}$ for all $j \in \{1,\dots,n\}$. Let $\tilde{\gamma} \in \mathbb{D}_n^+(\mathbb{R})$ be defined by $\tilde{\gamma}_{jj} \equiv a_{jj}^{-1}$ for all $j \in \{1,\dots,n\}$. Let f be as in (3). Let $g \in C^{m,\alpha}(\partial\Omega)$.*

Let $\tilde{u}$, $\tilde{c}$ be as in Theorem 3.3. Let ϵ', δ', $\tilde{\Gamma}$ be as in Theorem 3.4 (iii). Then the following statements hold.

(i) Let $s \in]0,+\infty[$. Let $\tilde{y} \in \mathbb{R}^n$. Let $J^\sharp$, $\mathcal{P}$ be as in Proposition 4.1. Then

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,l^{-1}s,\tilde{\gamma})}[u(\epsilon,l^{-1}s,\tilde{\gamma},\cdot)]\chi_{\tilde{y}+sQ}\,dx = s^nJ^\sharp[\epsilon,l^{-1}s,\tilde{\gamma}] + s^n(l^{-1}s)^2\int_Q \mathcal{P}[\tilde{\gamma}]\,dx$$

for all $l \in \mathbb{N}\setminus\{0\}$ such that $l > s/\delta'$ and for all $\epsilon \in]0,\epsilon'[$. Moreover

$$s^nJ^\sharp[0,0,\tilde{\gamma}] = s^nm_n(Q)\tilde{c} = \int_{\mathbb{R}^n}\tilde{c}\chi_{\tilde{y}+sQ}\,dx\,.$$

(ii) Let $\phi \in \mathcal{T}_q$. Let $s \in \mathbb{Q}\cap]0,+\infty[$ be as in Proposition 5.2 (ii). Then there exists a real analytic map H_ϕ from $]-\epsilon,\epsilon'[\times]-\delta',\delta'[\times\tilde{\Gamma}$ to $\mathbb{R}$ such that

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,l^{-1}s,\tilde{\gamma})}[u(\epsilon,l^{-1}s,\tilde{\gamma},\cdot)]\phi\,dx = s^nH_\phi[\epsilon,l^{-1}s,\tilde{\gamma}]$$

for all $l \in \mathbb{N}\setminus\{0\}$ such that $l > s/\delta'$ and for all $\epsilon \in]0,\epsilon'[$. Moreover,

$$s^nH_\phi[0,0,\tilde{\gamma}] = \int_{\mathbb{R}^n}\tilde{c}\phi(x)\,dx\,.$$

Proof. (i) Let $l \in \mathbb{N} \setminus \{0\}$ be such that $l > s/\delta'$. Since $u(\epsilon, l^{-1}s, \tilde{\gamma}, \cdot)$ is $l^{-1}s\tilde{\gamma}q$-periodic, it is also sq-periodic and accordingly,

$$\int_{\mathbb{R}^n} \mathbf{E}_{(\epsilon,l^{-1}s,\tilde{\gamma})}[u(\epsilon, l^{-1}s, \tilde{\gamma}, \cdot)](x)\chi_{\tilde{y}+sQ}(x)\, dx$$
$$= \int_{\tilde{y}+sQ} \mathbf{E}_{(\epsilon,l^{-1}s,\tilde{\gamma})}[u(\epsilon, l^{-1}s, \tilde{\gamma}, \cdot)](x)\, dx = \int_{sQ} \mathbf{E}_{(\epsilon,l^{-1}s,\tilde{\gamma})}[u(\epsilon, l^{-1}s, \tilde{\gamma}, \cdot)](x)\, dx\,.$$

Next we observe that

$$\bigcup_{0 \le z_j \le l a_{jj}-1} (qz + l^{-1}\tilde{\gamma}Q) \subseteq Q\,, \qquad m_n\left(Q \setminus \bigcup_{0 \le z_j \le l a_{jj}-1} (qz + l^{-1}\tilde{\gamma}Q)\right) = 0\,.$$

Accordingly, the $l^{-1}s\tilde{\gamma}q$-periodicity of $\mathbf{E}_{(\epsilon,l^{-1}s,\tilde{\gamma})}[u(\epsilon, l^{-1}s, \tilde{\gamma}, \cdot)]$ implies that

$$\begin{aligned}
&\int_{sQ} \mathbf{E}_{(\epsilon,l^{-1}s,\tilde{\gamma})}[u(\epsilon, l^{-1}s, \tilde{\gamma}, \cdot)](x)\, dx \\
&\quad = l^n \left(\Pi_{j=1}^n a_{jj}\right) \int_{l^{-1}s\tilde{\gamma}Q} \mathbf{E}_{(\epsilon,l^{-1}s,\tilde{\gamma})}[u(\epsilon, l^{-1}s, \tilde{\gamma}, \cdot)](x)\, dx \\
&\quad = l^n \left(\Pi_{j=1}^n a_{jj}\right) \int_{l^{-1}s\tilde{\gamma}(Q\setminus \mathrm{cl}\Omega_{p,\epsilon})} u(\epsilon, l^{-1}s, \tilde{\gamma}, x)\, dx \\
&\quad = l^n \left(\Pi_{j=1}^n a_{jj}\right) \int_{l^{-1}s\tilde{\gamma}(Q\setminus \mathrm{cl}\Omega_{p,\epsilon})} u^\sharp(\epsilon, l^{-1}s, \tilde{\gamma}, ls^{-1}\tilde{\gamma}^{-1}x)\, dx \\
&\quad = s^n \int_{Q\setminus \mathrm{cl}\Omega_{p,\epsilon}} u^\sharp(\epsilon, l^{-1}s, \tilde{\gamma}, y)\, dy = s^n J^\sharp[\epsilon, l^{-1}s, \tilde{\gamma}] + l^{-2}s^{n+2} \int_Q \mathcal{P}[\tilde{\gamma}]\, dx\,,
\end{aligned}$$

for all $l \in \mathbb{N} \setminus \{0\}$ such that $l > s/\delta'$ and for all $\epsilon \in]0, \epsilon'[$. Hence, statement (i) holds true.

Since ϕ is a finite linear combination of translations of functions such as $\chi_{\tilde{y}+sQ}$, statement (ii) is an immediate consequence of statement (i). □

6 A functional analytic representation theorem for the energy integral of the solution of problem (4)

We now turn to analyze the behavior of the energy integral of $u(\epsilon, \delta, \gamma, \cdot)$ in the periodic cell γQ as $(\epsilon, \delta, \gamma)$ approaches $(0, 0, \tilde{\gamma})$ in $]0, \epsilon_0[\times]0, +\infty[\times\mathbb{D}_n^+(\mathbb{R})$. We first introduce a notation for the energy integral of $u(\epsilon, \delta, \gamma, \cdot)$ in the periodic cell Q by means of the following.

Definition 6.1. *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0, 1[$. Let $p \in Q$. Let Ω be as in (1). Let ϵ_0 be as in (2). Let f be as in (3). Let $g \in C^{m,\alpha}(\partial\Omega)$. Then we set*

$$\mathrm{En}[\epsilon, \delta, \gamma] \equiv \int_{Q\cap\mathbb{S}(\epsilon,\delta,\gamma)^-} |D_x u(\epsilon, \delta, \gamma, x)|^2\, dx\,,$$

for all $(\epsilon, \delta, \gamma) \in]0, \epsilon_0[\times]0, +\infty[\times\mathbb{D}_n^+(\mathbb{R})$.

In the spirit of this paper, we now represent $\mathrm{En}[\epsilon, \delta, \gamma]$ in terms of analytic maps evaluated at triples $(\epsilon, \delta, \gamma)$ when δ and γ are such that the corresponding periodic cell $\delta\gamma Q$ is an integer fraction of the cell Q. In other words, we require that δ equals the reciprocal of some integer number $l \in \mathbb{N} \setminus \{0\}$ and that γ_{jj} equals the reciprocal of some integer number $a_{jj} \in \mathbb{N} \setminus \{0\}$ for all $j \in \{1, \dots, n\}$, and we prove the following.

Theorem 6.2. *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0, 1[$. Let $p \in Q$. Let Ω be as in (1). Let ϵ_0 be as in (2). Let $a_{jj} \in \mathbb{N}\setminus\{0\}$ for all $j \in \{1, \dots, n\}$. Let $\tilde{\gamma}_{jj} \equiv a_{jj}^{-1}$ for all $j \in \{1, \dots, n\}$. Let f be as in (3). Let $g \in C^{m,\alpha}(\partial\Omega)$. Let ϵ_e, $\mathcal{E}^\sharp$, $\mathcal{P}_e^\sharp$ be as in Proposition 4.2. Then there exists $l_e \in \mathbb{N} \setminus \{0\}$ such that*

$$\mathrm{En}[\epsilon, l^{-1}, \tilde{\gamma}] = l^2 \left\{\mathcal{E}^\sharp[\epsilon, l^{-1}, \tilde{\gamma}]\epsilon^{n-2} + l^{-4}\mathcal{P}_e^\sharp[\epsilon, \tilde{\gamma}]\right\}\,,$$

for all $\epsilon \in]0, \epsilon_e[$ and $l \in \mathbb{N} \setminus \{0\}$ such that $l \ge l_e$.

Proof. We first note that if $(\epsilon,\delta,\gamma)\in]0,\epsilon_e[\times]0,+\infty[\times\mathbb{D}_n^+(\mathbb{R})$, then we have

$$\begin{aligned}\int_{\delta\gamma(Q\setminus\mathrm{cl}\Omega_{p,\epsilon})}|D_xu(\epsilon,\delta,\gamma,x)|^2\,dx&=\int_{\delta\gamma(Q\setminus\mathrm{cl}\Omega_{p,\epsilon})}|D_x(u^\sharp(\epsilon,\delta,\gamma,\delta^{-1}\gamma^{-1}x))|^2\,dx\\&=\delta^{-2}\int_{\delta\gamma(Q\setminus\mathrm{cl}\Omega_{p,\epsilon})}|\gamma^{-1}D_xu^\sharp(\epsilon,\delta,\gamma,\delta^{-1}\gamma^{-1}x)|^2\,dx\\&=\delta^{n-2}\det\gamma\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}}|\gamma^{-1}D_xu^\sharp(\epsilon,\delta,\gamma,y)|^2\,dy\,.\end{aligned}\tag{46}$$

Next we note that if $\delta=l^{-1}$, $\gamma=\tilde\gamma$, then

$$Q\cap\left(\mathbb{R}^n\setminus\bigcup_{z\in\mathbb{Z}^n}\delta\gamma(qz+\mathrm{cl}\Omega_{p,\epsilon})\right)=Q\cap\left(\mathbb{R}^n\setminus\bigcup_{z\in\mathbb{Z}^n}l^{-1}\tilde\gamma(qz+\mathrm{cl}\Omega_{p,\epsilon})\right)\,,$$

and the set in the right hand side differs by a set of measure zero from the set

$$Q\cap\left(\bigcup_{z\in\mathbb{Z}^n}l^{-1}\tilde\gamma(qz+(Q\setminus\mathrm{cl}\Omega_{p,\epsilon}))\right)=\bigcup_{z\in\mathbb{Z}^n,\ 0\le z_j\le la_{jj}-1}\left(l^{-1}\tilde\gamma qz+l^{-1}\tilde\gamma(Q\setminus\mathrm{cl}\Omega_{p,\epsilon})\right)\,,$$

which is the union of a family of $l^n\left(\Pi_{j=1}^n a_{jj}\right)$ sets, all of which are a translation of $l^{-1}\tilde\gamma(Q\setminus\mathrm{cl}\Omega_{p,\epsilon})$. Hence, formula (46) implies that

$$\begin{aligned}&\int_{Q\cap\mathbb{S}(\epsilon,l^{-1},\tilde\gamma)^-}|D_xu(\epsilon,l^{-1},\tilde\gamma,x)|^2\,dx\\&\qquad=l^n\left(\Pi_{j=1}^n a_{jj}\right)\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}}|\tilde\gamma^{-1}D_xu^\sharp(\epsilon,l^{-1},\tilde\gamma,y)|^2\,dy l^{-(n-2)}\det\tilde\gamma\\&\qquad=l^2\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}}|\tilde\gamma^{-1}D_xu^\sharp(\epsilon,l^{-1},\tilde\gamma,y)|^2\,dy\qquad\forall l\in\mathbb{N}\setminus\{0\}\,.\end{aligned}$$

Then Proposition 4.2 implies that

$$\mathrm{En}[\epsilon,l^{-1},\tilde\gamma]=l^2\epsilon^{n-2}\mathcal{E}^\sharp[\epsilon,l^{-1},\tilde\gamma]+l^{-2}\mathcal{P}_e^\sharp[\epsilon,\tilde\gamma]\,,$$

for all $l\in\mathbb{N}\setminus\{0\}$ such that $l^{-1}<\delta'$, and for all $\epsilon\in]0,\epsilon_e[$. □

Next we want to show that Theorem 4.2 implies the validity of the following result, which could probably be deduced by variational techniques.

Proposition 6.3. *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $p\in Q$. Let Ω be as in (1). Let ϵ_0 be as in (2). Let $\tilde\gamma\in\mathbb{D}_n^+(\mathbb{R})$. Let f be as in (3). Let $g\in C^{m,\alpha}(\partial\Omega)$.*

Let $\tilde u$ be as in Theorem 3.3. Let $h\in]0,+\infty[$. If g is not a constant function and if $n\ge3$, then

$$\lim_{(\delta,\gamma)\to(0,\tilde\gamma)}\mathrm{En}[\delta^h,\delta,\gamma]=\begin{cases}0&\text{if }h>\frac{2}{n-2}\,,\\\int_{\mathbb{R}^n\setminus\mathrm{cl}\Omega}|\tilde\gamma^{-1}D\tilde u|^2\,dx&\text{if }h=\frac{2}{n-2}\,,\\+\infty&\text{if }h<\frac{2}{n-2}\,.\end{cases}$$

Proof. We first compute the number of elements of the set

$$Z^-(\delta\gamma)\equiv\{z\in\mathbb{Z}^n:\ \delta\gamma(qz+Q)\subseteq Q\}\,,$$

for each $(\delta,\gamma)\in]0,+\infty[\times\mathbb{D}_n^+(\mathbb{R})$. Let $(\delta,\gamma)\in]0,+\infty[\times\mathbb{D}_n^+(\mathbb{R})$. Clearly,

$$Z^-(\delta\gamma)=\left\{z\in\mathbb{Z}^n:\ 0\le z_j\le N_j^-(\delta\gamma)-1\quad\forall j\in\{1,\dots,n\}\right\}\,,$$

where $N_j^-(\delta\gamma)$ denotes the largest natural number such that

$$N_j^-(\delta\gamma)\delta\gamma_{jj}q_{jj}\le q_{jj}\,,$$

i.e., such that

$$N_j^-(\delta\gamma)\le\delta^{-1}\gamma_{jj}^{-1}\qquad\forall j\in\{1,\dots,n\}\,,$$

i.e.,

$$N_j^-(\delta\gamma) = [\delta^{-1}\gamma_{jj}^{-1}] \qquad \forall j \in \{1, \dots, n\} .$$

We also note that the number of elements of $Z^-(\delta\gamma)$ equals $\Pi_{j=1}^n [\delta^{-1}\gamma_{jj}^{-1}]$.

Next we fix $(\delta, \gamma) \in]0, +\infty[\times \mathbb{D}_n^+(\mathbb{R})$ and we compute the number of elements of the set

$$Z^+(\delta\gamma) \equiv \{z \in \mathbb{Z}^n : \delta\gamma(qz + Q) \cap Q \neq \emptyset\} .$$

Clearly,

$$Z^+(\delta\gamma) = \left\{z \in \mathbb{Z}^n : 0 \leq z_j \leq N_j^+(\delta\gamma) - 1 \ \ \forall j \in \{1, \dots, n\}\right\} ,$$

where $N_j^+(\delta\gamma)$ denotes the smallest natural number such that

$$q_{jj} \leq N_j^+(\delta\gamma)\delta\gamma_{jj}q_{jj} ,$$

i.e., such that

$$\delta^{-1}\gamma_{jj}^{-1} \leq N_j^+(\delta\gamma) \qquad \forall j \in \{1, \dots, n\} ,$$

i.e.,

$$N_j^+(\delta\gamma) = [\delta^{-1}\gamma_{jj}^{-1}]^- + 1 ,$$

where $[\delta^{-1}\gamma_{jj}^{-1}]^- = [\delta^{-1}\gamma_{jj}^{-1}]$ if $\delta^{-1}\gamma_{jj}^{-1} \in \mathbb{R} \setminus \mathbb{Z}$ and $[\delta^{-1}\gamma_{jj}^{-1}]^- = [\delta^{-1}\gamma_{jj}^{-1}] - 1$ if $\delta^{-1}\gamma_{jj}^{-1} \in \mathbb{Z}$. We note that the number of elements of $Z^+(\delta\gamma)$ equals $\Pi_{j=1}^n([\delta^{-1}\gamma_{jj}^{-1}]^- + 1)$. Moreover

$$\bigcup_{z \in Z^-(\delta\gamma)} \delta\gamma(qz + Q) \subseteq Q \subseteq \bigcup_{z \in Z^+(\delta\gamma)} \delta\gamma(qz + \mathrm{cl}Q) .$$

Next we note that the $\delta\gamma q$-periodicity of $u(\epsilon, \delta, \gamma, \cdot)$, and equality (8), and Proposition 4.2 imply that

$$\begin{aligned}
&\int_{Q\cap\mathbb{S}(\epsilon,\delta,\gamma)^-} |Du(\epsilon, \delta, \gamma, x)|^2 \, dx \qquad (47)\\
&\quad \geq \sum_{z \in Z^-(\delta\gamma)} \int_{\delta\gamma(qz+(Q\setminus\Omega_{p,\epsilon}))} |Du(\epsilon, \delta, \gamma, x)|^2 \, dx \\
&\quad = \left(\Pi_{j=1}^n [\delta^{-1}\gamma_{jj}^{-1}]\right) \int_{\delta\gamma(Q\setminus\Omega_{p,\epsilon})} |Du(\epsilon, \delta, \gamma, x)|^2 \, dx \\
&\quad = \left(\Pi_{j=1}^n [\delta^{-1}\gamma_{jj}^{-1}]\right) \int_{\delta\gamma(Q\setminus\Omega_{p,\epsilon})} |D_x(u^\sharp(\epsilon, \delta, \gamma, \delta^{-1}\gamma^{-1}x))|^2 \, dx \\
&\quad = \left(\Pi_{j=1}^n [\delta^{-1}\gamma_{jj}^{-1}]\right) \delta^{-2} \int_{\delta\gamma(Q\setminus\Omega_{p,\epsilon})} |\gamma^{-1}Du^\sharp(\epsilon, \delta, \gamma, \delta^{-1}\gamma^{-1}x)|^2 \, dx \\
&\quad = \left(\Pi_{j=1}^n [\delta^{-1}\gamma_{jj}^{-1}]\right) \det\gamma\delta^{n-2} \int_{Q\setminus\Omega_{p,\epsilon}} |\gamma^{-1}Du^\sharp(\epsilon, \delta, \gamma, y)|^2 \, dy \\
&\quad = \left(\Pi_{j=1}^n [\delta^{-1}\gamma_{jj}^{-1}]\right) (\Pi_{j=1}^n(\delta\gamma_{jj})) \left\{\epsilon^{n-2}\delta^{-2}\mathcal{E}^\sharp[\epsilon, \delta, \gamma] + \delta^2\mathcal{P}_e^\sharp[\epsilon, \gamma]\right\} ,
\end{aligned}$$

for all $(\epsilon, \delta, \gamma) \in]0, \epsilon'[\times]0, \delta'[\times\mathbb{D}_n^+(\mathbb{R})$. Similarly,

$$\begin{aligned}
&\int_{Q\cap\mathbb{S}(\epsilon,\delta,\gamma)^-} |Du(\epsilon, \delta, \gamma, x)|^2 \, dx \qquad (48)\\
&\quad \leq (\Pi_{j=1}^n(\delta\gamma_{jj})) \left(\Pi_{j=1}^n([\delta^{-1}\gamma_{jj}^{-1}]^- + 1)\right) \left\{\epsilon^{n-2}\delta^{-2}\mathcal{E}^\sharp[\epsilon, \delta, \gamma] + \delta^2\mathcal{P}_e^\sharp[\epsilon, \gamma]\right\} ,
\end{aligned}$$

for all $(\epsilon, \delta, \gamma) \in]0, \epsilon'[\times]0, \delta'[\times\mathbb{D}_n^+(\mathbb{R})$. Next we note that $\mathcal{E}^\sharp$ and $\mathcal{P}_e^\sharp$ are continuous, and that the assumption that g is not constant implies that

$$\mathcal{E}^\sharp[0, 0, \tilde{\gamma}] = \int_{\mathbb{R}^n} |\tilde{\gamma}^{-1}D\tilde{u}|^2 \, dx > 0 ,$$

and that

$$\lim_{(\delta,\gamma)\to(0,\tilde{\gamma})} \delta^{h(n-2)}\delta^{-2}\mathcal{E}^\sharp[\delta^h, \delta, \gamma] + \delta^2\mathcal{P}^\sharp[\delta^h, \gamma] = \begin{cases} 0 & \text{if } h > \frac{2}{n-2} , \\ \int_{\mathbb{R}^n\setminus\mathrm{cl}\Omega} |\tilde{\gamma}^{-1}D\tilde{u}|^2 \, dx & \text{if } h = \frac{2}{n-2} , \\ +\infty & \text{if } h < \frac{2}{n-2} . \end{cases}$$

Then by replacing ϵ by δ^h in (47), (48), the validity of the statement follows. □

We note that the criticality of the exponent $\frac{2}{n-2}$ has been observed a long time ago by Marčenko and Khruslov [39] and by Cioranescu and Murat [10, 11] for related problems (see also Maz'ya and Movchan [40], where the assumption of periodicity of the array of holes has been releaxed.)

A Auxiliary results

We first introduce the following result, which follows by standard properties of analytic functions in Roumieu classes and by (5) (cf. *e.g.*, [17, Prop. A.1].)

Theorem A.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\Omega \subseteq (\mathbb{R}^n \setminus q\mathbb{Z}^n)$. Let Γ be a bounded open subset of $\mathbb{D}_n^+(\mathbb{R})$ such that $\mathrm{cl}\Gamma \subseteq \mathbb{D}_n^+(\mathbb{R})$. Then there exists $\rho \in]0, +\infty[$ such that the map from Γ to $C^0_{\omega,\rho}(\mathrm{cl}\Omega)$ which takes γ to the function $S_{\gamma q,n}(\gamma\cdot)_{|\mathrm{cl}\Omega}$ is real analytic.*

Then we have the following. For a proof we refer to [33, § 5].

Theorem A.2. *The following statements hold.*

(i) The map from $\mathbb{D}_n^+(\mathbb{R}) \times (\mathbb{R}^n \setminus q(\mathbb{Z}^n \setminus \{0\}))$ to $\mathbb{R}$ which takes (γ, x) to $R_{\gamma q,n}(\gamma x)$ is real analytic.

(ii) Let Ω be a bounded open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\Omega \subseteq \mathbb{R}^n \setminus q(\mathbb{Z}^n \setminus \{0\})$. Let Γ be a bounded open subset of $\mathbb{D}_n^+(\mathbb{R})$ such that $\mathrm{cl}\Gamma \subseteq \mathbb{D}_n^+(\mathbb{R})$. Then there exists $\rho \in]0, +\infty[$ such that the map from Γ to $C^0_{\omega,\rho}(\mathrm{cl}\Omega)$ which takes γ to the function $R_{\gamma q,n}(\gamma\cdot)_{|\mathrm{cl}\Omega}$ is real analytic.

Next we introduce the following classes of singular functions in a periodically punctured domain.

Definition A.3. *Let $\lambda \in]0, +\infty[$. Let $\tilde{Q} \equiv \Pi_{l=1}^n] - q_{ll}/2, q_{ll}/2[$.*

(i) We denote by $A^0_{q,\lambda}$ the set of q-periodic elements h of $C^0(\mathbb{R}^n \setminus q\mathbb{Z}^n)$ such that

$$\sup_{x \in \tilde{Q} \setminus \{0\}} |h(x)|\,|x|^\lambda < +\infty\,,$$

and we set

$$\|h\|_{A^0_{q,\lambda}} \equiv \sup_{x \in \tilde{Q} \setminus \{0\}} |h(x)|\,|x|^\lambda \qquad \forall h \in A^0_{q,\lambda}\,.$$

(ii) We denote by $A^1_{q,\lambda}$ the set of q-periodic elements h of $C^1(\mathbb{R}^n \setminus q\mathbb{Z}^n)$ such that

$$h \in A^0_{q,\lambda}\,, \qquad \frac{\partial h}{\partial x_j} \in A^0_{q,\lambda+1} \qquad \forall j \in \{1, \dots, n\}\,,$$

and we set

$$\|h\|_{A^1_{q,\lambda}} \equiv \|h\|_{A^0_{q,\lambda}} + \sum_{j=1}^n \left\| \frac{\partial h}{\partial x_j} \right\|_{A^0_{q,\lambda+1}} \qquad \forall h \in A^1_{q,\lambda}\,.$$

One can readily verify that $(A^0_{q,\lambda}, \|\cdot\|_{A^0_{q,\lambda}})$ and $(A^1_{q,\lambda}, \|\cdot\|_{A^1_{q,\lambda}})$ are Banach spaces. Then we have the following result of [18]. For the convenience of the reader, we include a proof.

Proposition A.4. *Let $\rho \in]0, +\infty[$, $\lambda \in]0, n-1[$. The function $P_q[h, \varphi]$ from $\mathbb{R}^n$ to $\mathbb{R}$ defined by*

$$P_q[h, \varphi](x) \equiv \int_Q h(x-y)\varphi(y)\,dy \qquad \forall x \in \mathbb{R}^n\,,$$

belongs to $C^0_{q,\omega,\rho}(\mathbb{R}^n)$ for all $(h, \varphi) \in A^1_{q,\lambda} \times C^0_{q,\omega,\rho}(\mathbb{R}^n)$ and the operator from $A^1_{q,\lambda} \times C^0_{q,\omega,\rho}(\mathbb{R}^n)$ to $C^0_{q,\omega,\rho}(\mathbb{R}^n)$ which takes (h, φ) to $P_q[h, \varphi]$ is bilinear and continuous.

Proof. By exploiting the Divergence Theorem in a periodicity cell and the periodicity of h, φ, we can prove that

$$D^\alpha P_q[h, \varphi] = P_q[h, D^\alpha \varphi] \qquad \forall \alpha \in \mathbb{N}^n\,,$$

for all $(h, \varphi) \in A^1_{q,\lambda} \times C^0_{q,\omega,\rho}(\mathbb{R}^n)$. Now let $\mathbb{Z}_Q \equiv \{z \in \mathbb{Z}^n : qz \in \partial Q\}$. Obviously, $\mathbb{Z}_Q$ has 2^n elements, and $Q \subseteq \bigcup_{z \in \mathbb{Z}_Q}(qz + \mathrm{cl}\tilde{Q})$, and $m_n\left(Q \setminus \bigcup_{z \in \mathbb{Z}_Q}(qz + \tilde{Q})\right) = 0$. Then we have

$$\begin{aligned}
&\frac{\rho^{|\alpha|}}{|\alpha|!}\|D^\alpha P_q[h, \varphi]\|_{C^0(\mathrm{cl}Q)} \\
&\qquad = \frac{\rho^{|\alpha|}}{|\alpha|!}\|P_q[h, D^\alpha \varphi]\|_{C^0(\mathrm{cl}Q)} \leq \frac{\rho^{|\alpha|}}{|\alpha|!} 2^n \int_{\tilde{Q}} |y|^{-\lambda}\,dy \|h\|_{A^0_{q,\lambda}} \|D^\alpha \varphi\|_{C^0(\mathrm{cl}Q)} \\
&\qquad \leq 2^n \int_{\tilde{Q}} |y|^{-\lambda}\,dy \|h\|_{A^0_{q,\lambda}} \|\varphi\|_{C^0_{q,\omega,\rho}(\mathrm{cl}Q)} \qquad \forall \alpha \in \mathbb{N}^n\,.
\end{aligned}$$

Hence, the statement follows. □

Then we have the following variant of a result of Preciso [50, Prop. 1.1, p. 101].

Proposition A.5. *Let n_1, $n_2 \in \mathbb{N} \setminus \{0\}$, $\rho \in]0, +\infty[$, $m \in \mathbb{N}$, $\alpha \in]0, 1]$. Let Ω_1 be a bounded open subset of $\mathbb{R}^{n_1}$. Let Ω_2 be a bounded open connected subset of $\mathbb{R}^{n_2}$ of class C^1. Then the composition operator T from $C^0_{\omega,\rho}(\mathrm{cl}\Omega_1) \times C^{m,\alpha}(\mathrm{cl}\Omega_2, \Omega_1)$ to $C^{m,\alpha}(\mathrm{cl}\Omega_2)$ defined by*

$$T[u, v] \equiv u \circ v \quad \forall (u, v) \in C^0_{\omega,\rho}(\mathrm{cl}\Omega_1) \times C^{m,\alpha}(\mathrm{cl}\Omega_2, \Omega_1)\,,$$

is real analytic.

We also point out the validity of the following lemma on the fundamental solution.

Lemma A.6. *There exists $\varsigma \in]0, +\infty[$ such that*

$$\sup_{\xi \in \mathbb{R}^n \setminus \{0\}} |\xi|^{|\alpha|+(n-2)} |D^\alpha S_n(\xi)| \leq \varsigma^{|\alpha|} |\alpha|! \qquad \forall \alpha \in \mathbb{N}^n \setminus \{0\}\,.$$

Proof. Let $\alpha \in \mathbb{N}^n \setminus \{0\}$. Then there exist $\beta \in \mathbb{N}^n$ and $j \in \{1, \dots, n\}$ such that $\alpha = \beta + e_j$. If $\beta = 0$, then the inequality of the statement is satisfied with any $\varsigma \in [s_n^{-1}, +\infty[$. Thus we can assume that $\beta \neq 0$. Since $\frac{\partial S_n}{\partial \xi_j}$ is positively homogeneous of degree $-(n-1)$, then the Euler Theorem implies that $D^\beta \frac{\partial S_n}{\partial \xi_j}$ is positively homogeneous of degree $-(n-1) - |\beta|$. Since S_n is analytic in $\mathbb{R}^n \setminus \{0\}$ and $\partial \mathbb{B}_n(0, 1)$ is compact, then there exists $\varsigma \in [s_n^{-1}, +\infty[$ such that

$$\sup_{x \in \partial \mathbb{B}_n(0,1)} |D^\tau S_n(x)| \leq \varsigma^{|\tau|} \tau! \qquad \forall \tau \in \mathbb{N}^n \setminus \{0\}\,.$$

Hence, we conclude that

$$|D^\alpha S_n(\xi)| = |\xi|^{-(n-1)-|\beta|} \left| D^\beta \frac{\partial S_n}{\partial \xi_j} (\frac{\xi}{|\xi|}) \right| \leq |\xi|^{-(n-2)-|\alpha|} \varsigma^{|\alpha|} \alpha! \qquad \forall \xi \in \mathbb{R}^n \setminus \{0\}\,.$$

□

Then we have the following consequence of Theorem A.2 (ii).

Lemma A.7. *Let $\tilde{\gamma} \in \mathbb{D}_n^+(\mathbb{R})$. Then there exists an open neighborhood $\tilde{\Gamma}$ of $\tilde{\gamma}$ such that the map from $\tilde{\Gamma}$ to $A^1_{q,\max\{n-2,\frac{1}{2}\}}$ which takes γ to $S_{\gamma q,n} \circ \gamma$ is real analytic.*

Proof. Let $\lambda \equiv \max\{n-2, \frac{1}{2}\}$. Let

$$\begin{aligned} \tilde{A}^1_\lambda \equiv \Big\{ f \in C^1((\mathrm{cl}\tilde{Q}) \setminus \{0\}) : & \sup_{x \in \tilde{Q} \setminus \{0\}} |f(x)|\, |x|^\lambda < +\infty\,, \\ & \sup_{x \in \tilde{Q} \setminus \{0\}} |\frac{\partial f}{\partial x_j}(x)|\, |x|^{\lambda+1} < +\infty \ \forall j \in \{1, \dots, n\} \Big\} \\ \|f\|_{\tilde{A}^1_\lambda} \equiv \sup_{x \in \tilde{Q} \setminus \{0\}} |f(x)|\, |x|^\lambda & + \sum_{j=1}^n \sup_{x \in \tilde{Q} \setminus \{0\}} |\frac{\partial f}{\partial x_j}(x)|\, |x|^{\lambda+1} \qquad \forall f \in \tilde{A}^1_\lambda\,. \end{aligned}$$

Clearly, $(\tilde{A}^1_\lambda, \|\cdot\|_{\tilde{A}^1_\lambda})$ is a Banach space and the restriction map induces a linear isometry from $A^1_{q,\lambda}$ into $\tilde{A}^1_\lambda$. Hence, it suffices to show that there exists an open neighborhood $\tilde{\Gamma}$ of $\tilde{\gamma}$ in $\mathbb{D}_n^+(\mathbb{R})$ such that the map from $\tilde{\Gamma}$ to $\tilde{A}^1_\lambda$ which takes γ to $S_{\gamma q,n} \circ \gamma_{|(\mathrm{cl}\tilde{Q}) \setminus \{0\}} = S_n \circ \gamma_{|(\mathrm{cl}\tilde{Q}) \setminus \{0\}} + R_{\gamma q,n} \circ \gamma_{|(\mathrm{cl}\tilde{Q}) \setminus \{0\}}$ is real analytic.

By Theorem A.2 (ii), there exists a bounded open neighborhood $\tilde{\Gamma}$ of $\tilde{\gamma}$ in $\mathbb{D}_n^+(\mathbb{R})$ such that the map from $\tilde{\Gamma}$ to $C^1(\mathrm{cl}\tilde{Q})$ which takes γ to $R_{\gamma q,n} \circ \gamma_{|\mathrm{cl}\tilde{Q}}$ is real analytic. We can clearly assume that the closure of $\tilde{\Gamma}$ is compact and contained in $\mathbb{D}_n^+(\mathbb{R})$. Since the restriction map from $C^1(\mathrm{cl}\tilde{Q})$ to $\tilde{A}^1_\lambda$ is linear and continuous, the map from $\tilde{\Gamma}$ to $\tilde{A}^1_\lambda$ which takes γ to $R_{\gamma q,n} \circ \gamma_{|(\mathrm{cl}\tilde{Q}) \setminus \{0\}}$ is real analytic.

Next we turn to show that the map from $\tilde{\Gamma}$ to $\tilde{A}^1_\lambda$, which takes γ to $S_n \circ \gamma_{|(\mathrm{cl}\tilde{Q}) \setminus \{0\}}$ is real analytic. We first prove that there exist M, $r \in]0, +\infty[$ such that $D^\alpha_{(\gamma_{11}, \dots, \gamma_{nn})}(S_n(\gamma \cdot)_{|(\mathrm{cl}\tilde{Q}) \setminus \{0\}})$ belongs to $\tilde{A}^1_\lambda$ and

$$\|D^\alpha_{(\gamma_{11}, \dots, \gamma_{nn})}(S_n(\gamma \cdot)_{|(\mathrm{cl}\tilde{Q}) \setminus \{0\}})\|_{\tilde{A}^1_\lambda} \leq M \frac{|\alpha|!}{r^{|\alpha|}} \qquad \forall \alpha \in \mathbb{N}^n\,, \tag{49}$$

for all $\gamma \in \tilde{\Gamma}$. Since $S_n(\gamma x)$ is analytic in $(\gamma, x) \in \mathbb{D}_n^+(\mathbb{R}) \times (\mathbb{R}^n \setminus \{0\})$, we conclude that

$$D^\alpha_{(\gamma_{11},\dots,\gamma_{nn})}(S_n \circ \gamma_{|(\mathrm{cl}\tilde{Q})\setminus\{0\}}) \in C^1((\mathrm{cl}\tilde{Q}) \setminus \{0\}) \qquad \forall \gamma \in \mathbb{D}_n^+(\mathbb{R}) .$$

We also note that

$$\begin{aligned}
&|x| \leq \frac{\sqrt{n}}{2} \qquad \forall x \in \mathrm{cl}\tilde{Q} ,\\
&0 < g_1 \equiv \inf_{\gamma\in\tilde{\Gamma}} \min_{j=1,\dots,n} \gamma_{jj} < |\gamma| \leq \sup_{\gamma\in\tilde{\Gamma}} |\gamma| \equiv g_2 < \infty \qquad \forall \gamma \in \tilde{\Gamma} ,\\
&|\gamma x| \leq d_1 \equiv \frac{\sqrt{n}}{2} g_2 \qquad \forall (\gamma, x) \in \tilde{\Gamma} \times \mathrm{cl}\tilde{Q} .
\end{aligned}$$

By Lemma A.6, we have

$$\begin{aligned}
|D^\alpha_{(\gamma_{11},\dots,\gamma_{nn})}(S_n(\gamma x))|\,|x|^\lambda &= |x^\alpha D^\alpha S_n(\gamma x)|\,|x|^\lambda \\
&\leq |(\gamma_{11}^{-1},\dots,\gamma_{nn}^{-1})^\alpha (\gamma x)^\alpha D^\alpha S_n(\gamma x)|\,|\gamma x|^\lambda g_1^{-\lambda} \\
&\leq g_1^{-|\alpha|-\lambda} \left(\sup_{0<|\xi|\leq d_1} |\xi|^{|\alpha|} |D^\alpha S_n(\xi)|\,|\xi|^{n-2} \right) d_1^{\lambda-(n-2)} \\
&\leq g_1^{-|\alpha|-\lambda} \varsigma^{|\alpha|} |\alpha|! d_1^{\lambda-(n-2)} \qquad \forall (\gamma, x) \in \tilde{\Gamma} \times (\mathrm{cl}\tilde{Q}) \setminus \{0\})
\end{aligned} \tag{50}$$

for all $\alpha \in \mathbb{N}^n \setminus \{0\}$ and

$$\begin{aligned}
&\left| \frac{\partial}{\partial x_j} D^\alpha_{(\gamma_{11},\dots,\gamma_{nn})}(S_n(\gamma x)) \right| |x|^{\lambda+1} = \left| \frac{\partial}{\partial x_j} (x^\alpha D^\alpha S_n(\gamma x)) \right| |x|^{\lambda+1} \\
&\leq |\alpha_j x^{\alpha-e_j} D^\alpha S_n(\gamma x)|\,|x|^{\lambda+1} + |x^\alpha|\,|D^{\alpha+e_j} S_n(\gamma x)\gamma_{jj}|\,|x|^{\lambda+1} \\
&\leq |\alpha_j|\,|(\gamma_{11}^{-1},\dots,\gamma_{nn}^{-1})^{\alpha-e_j} (\gamma x)^{\alpha-e_j} D^\alpha S_n(\gamma x)|\,|\gamma x|^{(n-2)+1} |\gamma x|^{\lambda-(n-2)} g_1^{-\lambda-1} \\
&\quad + |(\gamma_{11}^{-1},\dots,\gamma_{nn}^{-1})^\alpha (\gamma x)^\alpha D^{\alpha+e_j} S_n(\gamma x)|\,|\gamma x|^{(n-2)+1} |\gamma x|^{\lambda-(n-2)} g_1^{-\lambda-1} g_2 \\
&\leq |\alpha_j| g_1^{-|\alpha|+1} \left(\sup_{0<|\xi|\leq d_1} |\xi|^{|\alpha|+(n-2)} |D^\alpha S_n(\xi)| \right) d_1^{\lambda-(n-2)} g_1^{-\lambda-1} \\
&\quad + g_1^{-|\alpha|-\lambda-1} g_2 \left(\sup_{0<|\xi|\leq d_1} |\xi|^{|\alpha|+1+(n-2)} |D^{\alpha+e_j} S_n(\xi)| \right) d_1^{\lambda-(n-2)} \\
&\leq |\alpha_j| g_1^{-|\alpha|+1} \varsigma^{|\alpha|} |\alpha|! d_1^{\lambda-(n-2)} g_1^{-\lambda-1} \\
&\quad + g_1^{-|\alpha|-\lambda-1} g_2 \varsigma^{|\alpha|+1} (|\alpha|+1)! d_1^{\lambda-(n-2)} \qquad \forall (\gamma, x) \in \tilde{\Gamma} \times ((\mathrm{cl}\tilde{Q}) \setminus \{0\})
\end{aligned} \tag{51}$$

for all $\alpha \in \mathbb{N}^n \setminus \{0\}$ and $j \in \{1, \dots, n\}$, where the first summand in the left hand side should be omitted if $\alpha_j = 0$. Since $(|\alpha| + 1) \leq 2^{|\alpha|+1}$ for all $\alpha \in \mathbb{N}^n \setminus \{0\}$, the inequalities (50) and (51) imply the existence of M and $r \in]0, +\infty[$ such that inequality (49) holds true. Possibly shrinking $\tilde{\Gamma}$, we can assume that $\tilde{\Gamma}$ is a ball of radius $r_1 \in]0, r[$. Then we introduce the map T from $\tilde{\Gamma}$ to $\tilde{A}^1_\lambda$ by setting

$$T[\gamma] \equiv \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!} D^\alpha_{(\gamma_{11},\dots,\gamma_{nn})}(S_n(\gamma\cdot)_{|(\mathrm{cl}\tilde{Q})\setminus\{0\}})(\gamma - \tilde{\gamma})^\alpha \qquad \forall \gamma \in \tilde{\Gamma} .$$

By inequality (49), the series which defines T is convergent in $\tilde{A}^1_\lambda$ uniformly in $\gamma \in \tilde{\Gamma}$ and accordingly T is real analytic. Since the convergence in $\tilde{A}^1_\lambda$ is stronger than the pointwise convergence in $(\mathrm{cl}\tilde{Q}) \setminus \{0\}$, we have

$$T[\gamma](x) = \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!} D^\alpha_{(\gamma_{11},\dots,\gamma_{nn})}(S_n(\gamma x))(\gamma - \tilde{\gamma})^\alpha \qquad \forall x \in (\mathrm{cl}\tilde{Q}) \setminus \{0\} , \tag{52}$$

for all $\gamma \in \tilde{\Gamma}$. Moreover, the function $S_n(\gamma x)$ is analytic in $(\gamma, x) \in \mathbb{D}_n^+(\mathbb{R}) \times (\mathbb{R}^n \setminus \{0\})$. Thus the right hand side of (52) equals $S_n(\gamma x)$ for all $(\gamma, x) \in \tilde{\Gamma} \times (\mathrm{cl}\tilde{Q}) \setminus \{0\})$. Hence, $T[\gamma] = S_n(\gamma\cdot)_{|(\mathrm{cl}\tilde{Q})\setminus\{0\}}$ for all $\gamma \in \tilde{\Gamma}$ and the map from $\tilde{\Gamma}$ to $\tilde{A}^1_\lambda$, which takes γ to $S_n(\gamma\cdot)_{|(\mathrm{cl}\tilde{Q})\setminus\{0\}}$ is analytic, and the proof is complete. □

Next, we have the following, which can be proved as Lemma 2.2 of [27].

Lemma A.8. *Let $\rho \in]0,+\infty[$. Let $p \in Q$. Let Ω be as in (1). Let ϵ_0 be as in (2). Let W be an open neighborhood of* $\mathrm{cl}Q$. *Then there exists a real analytic map G from $]-\epsilon_0,\epsilon_0[\times C^0_{\omega,\rho}(\mathrm{cl}W)$ to $\mathbb{R}$ such that*

$$\int_{Q\setminus\Omega_{p,\epsilon}} h\,dx = G[\epsilon,h] \qquad \forall(\epsilon,h)\in]0,\epsilon_0[\times C^0_{\omega,\rho}(\mathrm{cl}W)\,.$$

Finally, we introduce the following elementary lemma of [32, Lem. A.5].

Lemma A.9. *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$. Let $p \in Q$. Let Ω be as in (1). Let $\epsilon_0 \in]0,+\infty[$ be as in (2). Let $\epsilon_1 \in]0,\epsilon_0[$.*

(i) Let $\tilde{\Omega}$ be an open subset of $\mathbb{R}^n$ with a nonzero distance from $p+q\mathbb{Z}^n$. Then there exist $\epsilon^*_{\tilde{\Omega}} \in]0,\epsilon_1[$ such that

$$\mathrm{cl}\tilde{\Omega} \subseteq \mathbb{S}[\Omega_{p,\epsilon}]^- \qquad \forall\epsilon\in[-\epsilon^*_{\tilde{\Omega}},\epsilon^*_{\tilde{\Omega}}]\,,$$

and $\epsilon_{\tilde{\Omega}} \in]0,\epsilon^*_{\tilde{\Omega}}[$ such that

$$\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon^*_{\tilde{\Omega}}}]^- \subseteq \mathbb{S}[\Omega_{p,\epsilon}]^- \qquad \forall\epsilon\in[-\epsilon_{\tilde{\Omega}},\epsilon_{\tilde{\Omega}}]\,.$$

(ii) Let $\Omega^\sharp$ be a bounded open subset of $\mathbb{R}^n$ such that $\Omega^\sharp \subseteq \mathbb{R}^n\setminus\mathrm{cl}\Omega$. Then there exists $\epsilon_{\Omega^\sharp,r} \in]0,\epsilon_1[$ such that

$$p+\epsilon\mathrm{cl}\Omega^\sharp \subseteq Q\,, \qquad p+\epsilon\Omega^\sharp \subseteq \mathbb{S}[\Omega_{p,\epsilon}]^- \qquad \forall\epsilon\in[-\epsilon_{\Omega^\sharp,r},\epsilon_{\Omega^\sharp,r}]\setminus\{0\}\,.$$

Acknowledgement

This paper represents an extension of a part of the work performed by P. Musolino in his Doctoral Dissertation at the University of Padova under the guidance of M. Lanza de Cristoforis. The authors acknowledge the support of "Progetto di Ateneo: Singular perturbation problems for differential operators – CPDA120171/12" - University of Padova and the support of "INdAM GNAMPA Project 2015 - Un approccio funzionale analitico per problemi di perturbazione singolare e di omogeneizzazione". M. Lanza de Cristoforis acknowledges the support of the grant EP/M013545/1: "Mathematical Analysis of Boundary-Domain Integral Equations for Nonlinear PDEs" from the EPSRC, UK. P. Musolino has received funding from INdAM ("titolare di un assegno di ricerca INdAM"), from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 663830, and from the Welsh Government and Higher Education Funding Council for Wales through the Sêr Cymru National Research Network for Low Carbon, Energy and Environment.

References

[1] H. Ammari, J. Garnier, L. Giovangigli, W. Jing, and J.K. Seo, Spectroscopic imaging of a dilute cell suspension, J. Math. Pures Appl. (9) **105**, 603–661 (2016).

[2] H. Ammari, L. Giovangigli, H. Kwon, J.K. Seo, and T. Wintz, Spectroscopic conductivity imaging of a cell culture, Asymptot. Anal. **100**, 87–109 (2016).

[3] H. Ammari and H. Kang, Polarization and moment tensors (Springer, New York, 2007).

[4] N. Ansini and A. Braides, Asymptotic analysis of periodically-perforated nonlinear media, J. Math. Pures Appl. **81**, 439–451 (2002).

[5] V. Bonnaillie-Noël, M. Dambrine, S. Tordeux, and G. Vial, Interactions between moderately close inclusions for the Laplace equation, Math. Models Methods Appl. Sci. **19**, 1853–1882 (2009).

[6] A. Braides and A. Defranceschi, Homogenization of multiple integrals (The Clarendon Press, Oxford University Press, New York, 1998).

[7] L.P. Castro, D. Kapanadze, and E. Pesetskaya, A heat conduction problem of 2D unbounded composites with imperfect contact conditions, ZAMM Z. Angew. Math. Mech. **95**, 952–965 (2015).

[8] L.P. Castro, D. Kapanadze, and E. Pesetskaya, Effective conductivity of a composite material with stiff imperfect contact conditions, Math. Methods Appl. Sci. **38**, 4638–4649 (2015).

[9] L. P. Castro, E. Pesetskaya, and S. V. Rogosin, Effective conductivity of a composite material with non-ideal contact conditions, Complex Var. Elliptic Equ. **54**, 1085–1100 (2009).

[10] D. Cioranescu and F. Murat, Un terme étrange venu d'ailleurs, in Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. II (Paris, 1979/1980), vol. 60 of Res. Notes in Math. (Pitman, Boston, Mass.-London, 1982), pp. 98–138, 389–390.

[11] D. Cioranescu and F. Murat, Un terme étrange venu d'ailleurs. II, in Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. III (Paris, 1980/1981), vol. 70 of Res. Notes in Math. (Pitman, Boston, Mass.-London, 1982), pp. 154–178, 425–426.

[12] C. Conca, D. Gómez, M. Lobo, and M.E. Pérez, Homogenization of periodically perforate media, Indiana Univ. Math. J. **48**, 1447–1470 (1999).

[13] G. Dal Maso and F. Murat, Asymptotic behaviour and correctors for linear Dirichlet problems with simultaneously varying operators and domains, Ann. Inst. H. Poincaré Anal. Non Linéaire **21**, 445–486 (2004).

[14] M. Dalla Riva, Stokes flow in a singularly perturbed exterior domain, Complex Var. Elliptic Equ. **58**, 231–257 (2013).

[15] M. Dalla Riva and M. Lanza de Cristoforis, Microscopically weakly singularly perturbed loads for a nonlinear traction boundary value problem: a functional analytic approach, Complex Var. Elliptic Equ. **55**, 771–794 (2010).

[16] M. Dalla Riva and M. Lanza de Cristoforis, A singularly perturbed nonlinear traction boundary value problem for linearized elastostatics. A functional analytic approach, Analysis (Munich) **30**, 67–92 (2010).

[17] M. Dalla Riva, M. Lanza de Cristoforis, and P. Musolino, Analytic dependence of volume potentials corresponding to parametric families of fundamental solution, Integral Equations Operator Theory **82**, 371–393 (2015).

[18] M. Dalla Riva, M. Lanza de Cristoforis, and P. Musolino, Mapping properties of weakly singular periodic volume potentials in Roumieu classes, Submitted.

[19] K. Deimling, Nonlinear functional analysis (Springer-Verlag, Berlin, 1985).

[20] P. Drygaś and V. Mityushev, Effective conductivity of unidirectional cylinders with interfacial resistance, Quart. J. Mech. Appl. Math. **62**, 235–262 (2009).

[21] G. B. Folland, Introduction to partial differential equations (Princeton University Press, Princeton, NJ, 1995).

[22] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order (Springer-Verlag, Berlin, 1983).

[23] V. Kozlov, V. Maz'ya, and A. Movchan, Asymptotic analysis of fields in multi-structures (The Clarendon Press, Oxford University Press, New York, 1999). Oxford Science Publications.

[24] M. Lanza de Cristoforis, Properties and pathologies of the composition and inversion operators in Schauder spaces, Rend. Accad. Naz. Sci. XL Mem. Mat. **15**, 93–109 (1991).

[25] M. Lanza de Cristoforis, Asymptotic behaviour of the conformal representation of a Jordan domain with a small hole in Schauder spaces, Comput. Methods Funct. Theory **2**, 1–27 (2002).

[26] M. Lanza de Cristoforis, Asymptotic behaviour of the conformal representation of a Jordan domain with a small hole, and relative capacity, in Complex analysis and dynamical systems, vol. 364 of Contemp. Math. (Amer. Math. Soc., Providence, RI, 2004), pp. 155–167.

[27] M. Lanza de Cristoforis, A singular domain perturbation problem for the Poisson equation, in More progresses in analysis. Proceedings of the 5th international ISAAC congress, Catania, Italy, July 25-30 (World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005), pp. 995–965.

[28] M. Lanza de Cristoforis, Asymptotic behavior of the solutions of a nonlinear Robin problem for the Laplace operator in a domain with a small hole: a functional analytic approach, Complex Var. Elliptic Equ. **52**, 945–977 (2007).

[29] M. Lanza de Cristoforis, Asymptotic behavior of the solutions of the Dirichlet problem for the Laplace operator in a domain with a small hole. A functional analytic approach, Analysis (Munich) **28**, 63–93 (2008).

[30] M. Lanza de Cristoforis and P. Musolino, A perturbation result for periodic layer potentials of general second order differential operators with constant coefficients, Far East J. Math. Sci. (FJMS) **52**, 75–120 (2011).

[31] M. Lanza de Cristoforis and P. Musolino, A real analyticity result for a nonlinear integral operator, J. Integral Equations Appl. **25**, 21–46 (2013).

[32] M. Lanza de Cristoforis and P. Musolino, A singularly perturbed nonlinear Robin problem in a periodically perforated domain: a functional analytic approach, Complex Var. Elliptic Equ. **58**, 511–536 (2013).

[33] M. Lanza de Cristoforis and P. Musolino, Analytic dependence of a periodic analog of a fundamental solution upon the periodicity parameters, Submitted, (2016).

[34] M. Lanza de Cristoforis and P. Musolino, A functional analytic approach to homogenization problems, in Integral methods in science and engineering. Theoretical and computational advances (Cham: Birkhäuser/Springer, 2015), pp. 269–289.

[35] M. Lanza de Cristoforis and L. Rossi, Real analytic dependence of simple and double layer potentials upon perturbation of the support and of the density, J. Integral Equations Appl. **16**, 137–174 (2004).

[36] M. Lanza de Cristoforis and L. Rossi, Real analytic dependence of simple and double layer potentials for the Helmholtz equation upon perturbation of the support and of the density, in Analytic methods of analysis and differential equations: AMADE 2006 (Camb. Sci. Publ., Cambridge, 2008), pp. 193–220.

[37] C.S. Lin and C.L. Wang, Elliptic functions, Green functions and the mean field equations on tori, Ann. of Math. **172**, 911–954 (2010).

[38] M. Mamode, Fundamental solution of the Laplacian on flat tori and boundary value problems for the planar Poisson equation in rectangles, Bound. Value Probl., **221**, 1–9 (2014).

[39] V. A. Marčenko and E. Y. Khruslov, Kraevye zadachi v oblastyakh s melkozernistoi granitsei (Izdat. "Naukova Dumka", Kiev, 1974).

[40] V. Maz'ya and A. Movchan, Asymptotic treatment of perforated domains without homogenization, Math. Nachr. **283**, 104–125 (2010).

[41] V. Maz'ya, A. Movchan, and M. Nieves, Green's kernels and meso-scale approximations in perforated domains (Springer, Heidelberg, 2013).

[42] V. Maz'ya, S. Nazarov, and B. Plamenevskij, Asymptotic theory of elliptic boundary value problems in singularly perturbed domains. Vol. I, vol. 111 of Operator Theory: Advances and Applications (Birkhäuser Verlag, Basel, 2000).

[43] V. Maz'ya, S. Nazarov, and B. Plamenevskij, Asymptotic theory of elliptic boundary value problems in singularly perturbed domains. Vol. II, vol. 112 of Operator Theory: Advances and Applications (Birkhäuser Verlag, Basel, 2000).

[44] C. Miranda, Sulle proprietà di regolarità di certe trasformazioni integrali, Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. I, **7**, 303–336 (1965).

[45] V. Mityushev and P. M. Adler, Longitudinal permeability of spatially periodic rectangular arrays of circular cylinders. I. A single cylinder in the unit cell, ZAMM Z. Angew. Math. Mech. **82**, 335–345 (2002).

[46] P. Musolino, A singularly perturbed Dirichlet problem for the Laplace operator in a periodically perforated domain. A functional analytic approach, Math. Methods Appl. Sci. **35**, 334–349 (2012).

[47] P. Musolino, A singularly perturbed Dirichlet problem for the Poisson equation in a periodically perforated domain. A functional analytic approach, in Advances in harmonic analysis and operator theory, vol. 229 of Oper. Theory Adv. Appl. (Birkhäuser/Springer Basel AG, Basel, 2013), pp. 269–289.

[48] S. A. Nazarov and J. Sokołowski, Asymptotic analysis of shape functionals, J. Math. Pures Appl., **82**, 125–196 (2003).

[49] S. Ozawa, Electrostatic capacity and eigenvalues of the Laplacian, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **30**, 53–62 (1983).

[50] L. Preciso, Regularity of the composition and of the inversion operator and per- turbation analysis of the conformal sewing problem in Romieu type spaces, Tr. Inst. Mat. Minsk **5**, 99–104 (2000).

[51] S. Rogosin, M. Dubatovskaya, and E. Pesetskaya, Eisenstein sums and functions and their application at the study of heat conduction in composites, Šiauliai Math. Semin. **4**, 167–187 (2009).

[52] M. J. Ward and J. B. Keller, Strong localized perturbations of eigenvalue problems, SIAM J. Appl. Math. **53**, 770–798 (1993).